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This curriculum bulletin is one of a planned series of bulletins designed to meet the needs of teachers and supervisors who are working to improve the achievement level of mathematics. The material has been planned to help teachers meet the diverse mathematical needs of the children in fifth-grade classes. In addition to the emphasis that is always placed on arithmetic computational skills, this bulletin shows how to include other areas considered important, such as concepts, skills, and ideas from algebra and geometry. The 80 units of this bulletin are organized into three categories: sets, number, numeration; operations; and geometry and measurement. The units are sequentially planned and follow a spiral pattern. Objectives for each unit are stated. This is the first part of a two part bulletin for Grade 5. (RP)

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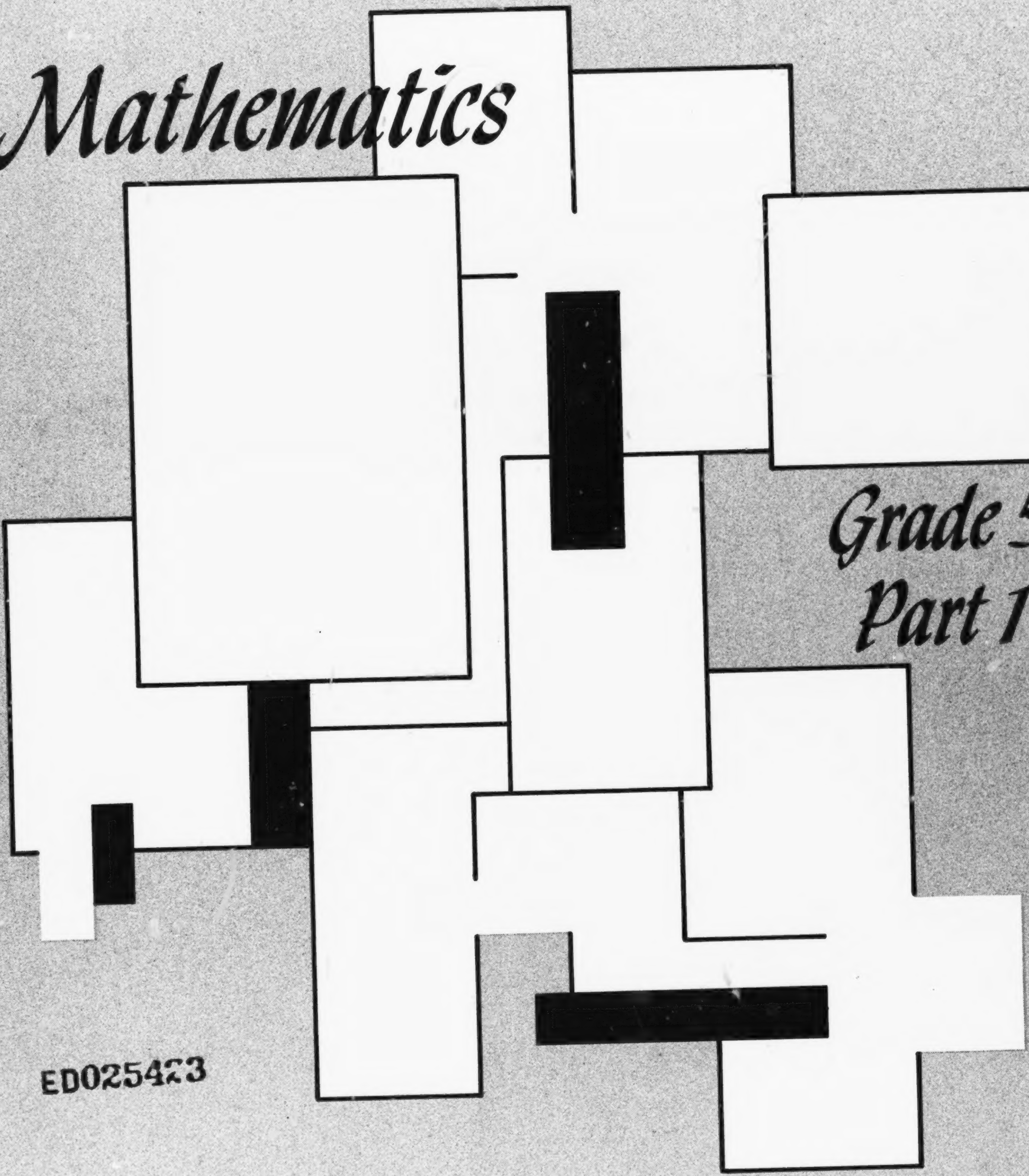
Mathematics



Grade 5
Part 1

ED025423

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Board of Education of the City of New York

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Mathematics
Grade 5
Part 1

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FOREWORD

The MATHEMATICS, GRADE 5, CURRICULUM BULLETIN is one of a planned series of bulletins designed to meet the needs of teachers and supervisors who are working to improve the achievement level of mathematics in our schools. The material has been planned to help teachers meet the diverse mathematical needs of the children in fifth-grade classes in our schools. In addition to the emphasis that is always placed on arithmetic computational skills, this bulletin shows how to include other areas considered important, such as, concepts, skills, and ideas from Algebra and Geometry.

Use of the new bulletin will provide articulation with grade 4 mathematics and with the mathematics of grades 6, 7, and 8. The publication completes a four-year sequence in Intermediate School Mathematics based on the newer mathematical philosophy of what should and can be taught to children in grades 5 through 8. Revisions will be made in this publication as a result of use in schools during the introductory period.

ACKNOWLEDGMENTS

The impetus for Mathematics Grade 5 was provided by Mr. George Grossman, Acting Director of the Bureau of Mathematics and by Mrs. Ella Simpson, Acting Assistant Director of the Bureau of Mathematics who recognized the need for a more modern approach to the teaching of mathematics in Grade 5.

Dr. Joseph O. Loretan,* Deputy Superintendent of Schools and Mrs. Helene Lloyd, Assistant Superintendent supervised the project.

Assistant Superintendent Dr. William Bristow, Director of The Bureau of Curriculum Research and members of his Bureau cooperated in the initial discussions.

Mrs. Leona Critchley and Mrs. Alice Lombardi, Staff Coordinators of the Bureau of Mathematics, planned and prepared the sequence and organization, wrote the units, and edited the final draft for publication.

Mrs. Eva Pollack, Staff Coordinator of the Bureau of Mathematics, reviewed the units, made suggestions, and rewrote parts of some units.

Mr. Grossman, as Acting Director of Mathematics edited and suggested revisions and additions.

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Miss Anne Piccini prepared the diagrams and designed the cover. Mr. Albert Lacerre and Mr. Maurice Basseches facilitated the printing.

Grateful acknowledgment is made to all those responsible for the production of Mathematics Cycles Grade 5 and Mathematics Cycles Grade 6.

**Deceased*

INTRODUCTION

This is the first part of a two part bulletin that has been prepared as a revision of the earlier Mathematics 5 Cycles. It includes all of the topics in Mathematics Cycles — Grade 5(1) enriched and expanded to include the newer emphasis of a modern mathematics curriculum as found in the 1965 edition of the Mathematics 6 Bulletin. All relevant material in these bulletins has been utilized.

This will mean that students completing this course will be in a better position to complete a more thorough course in Mathematics 6.

It *is* important for children to develop speed and accuracy in computation, but in addition to computational skills, it is important for children to develop understanding not only of arithmetic concepts but of concepts from algebra and geometry.

The 80 units of this bulletin are organized into 3 categories:

Sets; Number; Numeration
Operations
Geometry and Measurement

These categories are shown in the Scope and Sequence Chart on page xx. This chart may also be considered a Table of Contents.

The 80 units are sequentially planned. For example: After an introduction to sets, under the category "Sets," children are led to see the relation between Union of Sets and Addition of Whole Numbers under the category "Operations." This pattern is followed throughout.

The units also follow a *spiral* pattern, in that development of concepts and operations are repeated at increasing levels of understanding.

Concepts from Algebra, such as: open sentences, relations between numbers, graphing of solution sets, are included in the exercises of most of the units. Concepts from geometry are also included. A "Note to Teacher" is included in those units where it was felt the teacher might want further clarification of the mathematical concepts connected with the unit and/or to understand reasons for the developmental material.

Objectives for each unit are clearly stated immediately before the "Teaching Suggestions" designed for the implementation of those objectives.

Review of background necessary for the introduction of new topics is suggested where necessary. For example: Before adding and subtracting fractional numbers using the least common denominator method, suggestions are made for renaming fractions, regrouping fractions, etc.

Asterisks before a unit or before an item within that unit indicate that these developments may be used at the discretion of the teacher.

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(1) Mathematics Cycles, Grades 5 — Curriculum Bulletin. 1961-62 Series, No. 6

SCOPE AND SEQUENCE

Pages vi to xi

Grade 5

SCOPE AND SEQUENCE

* — indicates optional units
page numbers below — indicate location of units

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SCOPE AND SEQUENCE

* — indicates optional units
page numbers below — indicate location of units

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SCOPE AND SEQUENCE

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SETS; NUMBER; NUMERATION

UNIT 1 - SETS

NOTE TO TEACHER

Why Sets?

A set may be defined as a collection or group of objects, ideas, or numbers.

The use of sets of concrete objects provides a visual and tactile experience for the development of the abstract concept of "number" and for operations on number.

Since geometric and algebraic, as well as arithmetic ideas, can be expressed in terms of sets, set concepts and terminology have a wide application.

Meaning of Set

A mathematical name for a collection of things is "Set".

A set may be composed of similar or dissimilar objects, things or ideas, which we decide to consider as a unit.

All the mathematics books that we use may be similar, but each is a different, discrete book. The collection of Mathematics Books can be considered as a set.

A collection of completely dissimilar things such as a pencil, a stapler, an orange, may also be considered as a set. These may be the set of objects on the Teacher's desk.

To convey the idea that a collection is a set, the collection must be clearly defined.

A collection of "Interesting Books" is not a clearly defined set because what is interesting to one may not be interesting to another. The set "Mathematics Books in our Classroom" is clearly defined.

Elements of a Set

Each object, thing or idea in a set is called a member or an element of that set. For example, the elements of a set of golf clubs are the individual clubs in that set.

Describing the set clearly helps to determine whether an object is or is not an element of the set.

For example, when we talk about, "The Set of The Five Boroughs of New York City", there is no doubt that Richmond is an element of that set.

We may also describe this set by tabulating its elements, for example, "The Set Whose Elements are: Manhattan, Bronx, Brooklyn, Queens, Richmond".

A set may consist of many elements. The set of all teachers in New York City is a set containing many elements.

A set may consist of only one member, for example, "The Set of the Teachers in this Room".

A set may contain no members, for example, "The Set of All Teachers Who Are 15 Feet Tall".

Notation for Sets

The elements of a set may be tabulated within braces. e.g., {Debbie, Gail, Kathy} can be read as: "The set whose elements are Debbie, Gail, Kathy".

It should be understood that when we list the elements of a set we never list the same element more than once.

{ Δ, □ } can be read as "The set whose elements are triangle, square".

Sets may also be identified by the use of a capital letter.

For example:

A = {Debbie, Gail, Kathy} read as: "Set A is the set whose elements are Debbie, Gail, Kathy".

B = {Bill} read as: "Set B is the set whose only member is Bill".

C = {apple, banana} can be read as: "Set C is the set whose elements are apple, banana".

When the sets are referred to again they may be recorded as: A, B, and C.

The empty set can be designated by braces only, { } , or by the symbol ∅ . The empty set is sometimes called the null set.

The "belonging to" symbol for showing that an object is an element of a set is ∈ . The statement, "Kathy ∈ A", is read: Kathy is an element of Set A".

The symbol for an element that is not in a set is ∉ . The statement, "Ann ∉ A", is read: "Ann is not an element of Set A".

TEACHING SUGGESTIONS

Objective To help children understand the meaning of: Elements of a Set; Sets; Subset; Set Notation.

Procedure The previous exposure of the children to these concepts will help the teacher determine the extent to which she will use the following procedures.

1. Begin if necessary by presenting physical objects such as: a set of dishes; a set of checkers; etc. Elicit that a collection of things is called a set. Tell children that the mathematical name for a collection of objects is "Set".

2. Display and discuss sets of objects. Include some examples in which the elements of the set are similar, for example: two crayons; some in which the elements are dissimilar, for example; a book, a ball, and a piece of chalk.

   Children discuss dissimilar objects as a set.

3. Have children use available objects to create their own sets on their desks and then to describe their sets.

   Have them describe sets of similar things. Make sure they understand that each thing is a discrete object.

4. Children follow the same procedure using sets of dissimilar objects. They discuss why these are considered as sets.

5. Discuss sets that are not easily displayed.
   (The set of characters in a book; the set of states in the U.S.)

6. Tell children that each object in a set is called a member or an element of the set. Have children name elements in displayed sets; in sets of similar objects; in sets of dissimilar objects.

7. Tell children that a set may have only one element. Have them think of and discuss sets that have only one element.

   [the class aquarium; the principal of the school;
   the capital building of New York State.]

8. Ask the set of tigers in the room to stand. When no one stands ask:

   Is there a set of tigers? [Yes]
   Is there a set of tigers in this room? [No, the set of tigers is the empty set].

9. Tell children that a set may have no elements and that this kind of set is called the empty set, or the null set. Have children suggest sets which contain no elements.

   [The set of women who have been presidents of the United States;
   The set of living dinossaurs; The set of astronauts in Grade 5.]

10. Discuss the use of symbols as a way of conveying ideas.

For example, Highway signs are a set of symbols that communicate ideas of curve, intersecting roads, etc.

↷ ⊥ +

11. Introduce set symbols.

A way of specifying a set is to list the names of the elements or symbols for the elements between braces { }.

{a piece of chalk, a globe, a milk container}

{ □, Δ, * , 0 } is read as: the set whose elements are a square, triangle, star, circle.

Make sure that children understand that the same set may be recorded in many different ways, for example:

{ Δ, * , □ , 0 }, { 0, □, Δ , * }

or

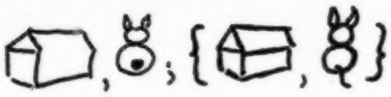
{The living presidents of the U.S.}

or

{L:B: Johnson, Harry Truman, Eisenhower}

Tell children that the order of listing the members of a set does not matter. Children should note that elements in a set are separated by commas. The same element appears only once in a listing.

12. Record on the board:

Mary, Jane; {Jane, Mary} 

Have children identify those that are sets.
[Only those within braces are sets.]

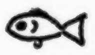
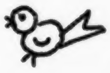
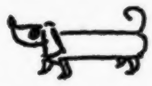
13. Tell children that a set of many objects can be thought of as a single idea and may be named by a capital letter, e.g., A, B, C.

Record: A = {Bob, Sam, Jack}

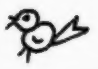
Read to children as: Set A is the set whose elements are Bob, Sam, Jack.

Present other sets in the same way. Have children write their own sets using capital letters.

14. Introduce the symbol for element of a set ($\in$).

Record: A = { , ,  }

Children read: "A is the set whose elements are a fish, a bird, a dog".

Record:  $\in$ A

Read to children as: "A bird is an element of set A".

PRACTICE and / or EVALUATION
SUGGESTED EXERCISES

1. Name as many words as you can that mean set.

   [Collection, group, team, etc.]

2. Write the set of odd numbers from 2 through 9.

   [$\{3, 5, 7, 9\}$]

3. Record next to the following sets, using set notation, those that have no elements:

   The set of dogs that fly.
   The set of children in your fifth grade who have green hair.
   The set of dishes in the closet.
   The set of letters of the alphabet.
   The set of even whole numbers less than 1.

4. List the elements of the following sets:

   The set of Great Lakes.
   The set of children who are officers of your class.
   The set of the capital city of New York State.

5. Record in set notation two examples of sets containing more than one element; two examples of sets containing only one element; two examples of sets containing no elements.

6. Write the following statement in set notation:

   Set A is the empty set.

   [A = { }]

7. Consider the following sets:

   A = {John, Mary, baby-Sue}

   F = {Monday, Tuesday, Wednesday, Thursday, Friday}

   S = { ▭ , ✱ , Δ }

   Write these sentences in set notation:

   John is an element of set A.
   Tuesday is an element of set F.
   A rectangle is not a member of set A.
   Baby-Sue belongs to set A.
   A triangle is an element of set S.

8. Use set notation to write:

   The set of letters in your name.
   The set of numbers between 20 and 30.
   The set of numbers that are less than 10.

SETS; NUMBER; NUMERATION

UNIT 2- SETS: One-to-One Correspondence; Equivalent Sets; Number as a Property of a Set; The Set of Counting Numbers

NOTE TO TEACHER

One-to-One Correspondence

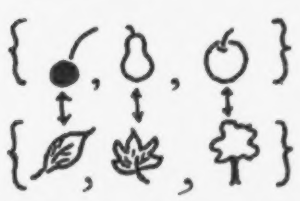
Matching objects in one set to objects in another set is not new. Children have always been encouraged to note when there is one book for each child, a chair for each child, a glove for each hand.

When the elements of two sets can be paired so that each element of one set is associated with one and only one element of the other set, and no element of either set is excluded in this pairing, the two sets are said to be in one-to-one correspondence.

The elements of the two sets that are being matched may or may not be similar.

The sets below can be matched in a one-to-one correspondence.

The order in which they are matched is irrelevant.



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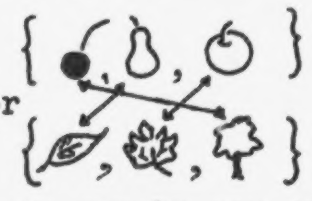

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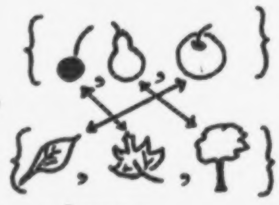
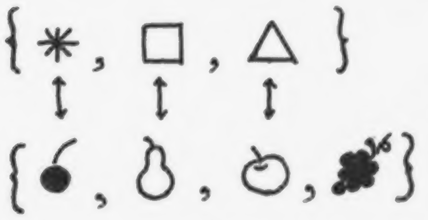


These sets are not in one-to-one correspondence.



$\{a, b, c, d\}$

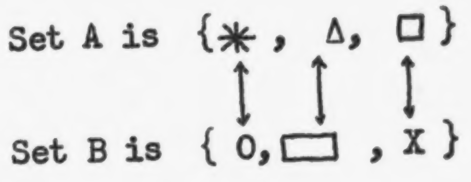
$\{x, y\}$

Equivalent Sets

Two sets of elements that can be placed in one-to-one correspondence are said to be equivalent sets.

Equivalent sets have the same number of elements.

The elements of the sets below have been put into one-to-one correspondence.



Set A is {✳, Δ, □}

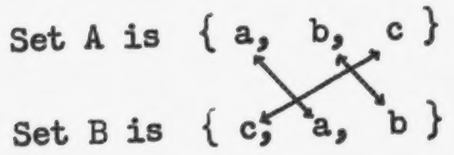
Set B is {O, ▭, X}

Each element of Set A is paired with one and only one element of Set B in each instance and each element of Set B is paired with one and only one element of Set A.

Set A is equivalent to Set B

Equal Sets

When two sets have the same elements and therefore have the same number of elements, they are said to be equal sets.



Set A is {a, b, c}

Set B is {c, a, b}

The elements of Set A and Set B are the same.

Every element of Set A is an element of Set B.

Every element of Set B is an element of Set A.

The number of elements in Set A and in Set B is the same.

Therefore:

Set A equals Set B

Set A and Set B are different names for the same set, (which is the usual meaning of "equal" in mathematical situations).

When we compare sets we find that one set may be equivalent to another set; one set may be equal to another set; one set may have more or fewer elements. In the last case it is neither equal to nor equivalent to the other set.

Number as a Property of a Set

Consider a set which contains 5 green objects. Its elements have the color property of "greeness".

The set has a number property of "five-ness".

When sets can be placed in a one-to-one correspondence, they have the same number property.

$\{ a, b, c, d \}$ has a number property of "4-ness".

$\{ x, y, r, s \}$ has a number property of "4-ness".

Both sets have the same number property.
The numbers are equal.

When two sets can not be placed in a one-to-one correspondence, they do not have the same number property.

One number is greater than or is less than the other.

Mathematical symbols for inequalities of number are:

"Is greater than", " > ", for example:

$7 > 5$; $5 - 2 > 6 - 4$

"Is less than", " < ", for example:

$2 < 3$; $3 + 8 < 4 + 9$

The Set of Counting Numbers

When children began to count, they began with one object. They associated the number 1 with a set containing one object; the number 2 with a set containing two objects and so on.

The set of numbers used in counting beginning with 1 is called the, "Set of Counting Numbers" and is shown in set notation as $\{1, 2, 3, 4, 5, \ldots\}$

Children should understand that no matter how far they count there is still another number. This is indicated by three dots as shown in the set above, standing for "and so forth".

TEACHING SUGGESTIONS

Objectives: To reinforce the concepts of one-to-one correspondence.
To develop number as a property of sets.

Procedure

1. Question children about the common property of familar things.

   What property do rain, milk and the ocean have in common?
   [liquid]

   What property do a cookie, a chocolate bar, a piece of pie have in common?
   [sweet, edible, fattening]

2. Compare sets of things in the classroom where a 1 - 1 matching is obvious.
   One pencil for one child
   One sheet of construction paper for each child
   One plastic spoon for each jar of paint

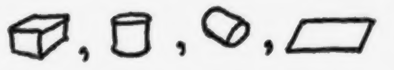
3. Discuss these matchings with the children.

4. Show two sets of dissimilar objects that are in one-to-one correspondence on a display table, for example:
   { a book, a pencil, a paint jar }
   { a ball, a bat, a stapler }

   Discuss the dissimilarity of the elements of the sets.

5. Ask children: What property do the sets on the display table have in common?
   [The number 3]

6. Compare other sets of objects that are in one-to-one correspondence to emphasize that number is a property of set. For example:

   a. { O, Δ, ✳, x } and {  } [4]

   b. The number of children in Grade Five who are 3 years old.
      [{ }]

      and

      The number of children in Grade One who are 25 years old.
      [{ }]

7. Help children to see that sets that can be placed in one-to-one correspondence have the same number.

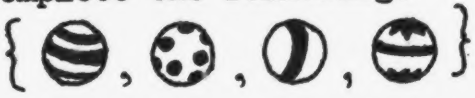
8. Tell children:

   a. A way we indicate the number of elements in a set is by using the letter N (for number) before the set. For example:

      N { ✳, Δ, O }

   b. When we write N { ✳ , Δ, O } = 3 we are saying that the number of elements in the set whose members are star, triangle, circle, is 3.

9. Children complete the following:

   A = {  }

   Number of elements = ☐ [4]
   N (A) = ☐ [4]

   G = {Sue, Betty, Jane}

   Number of elements = ☐ [3]
   N (G) = ☐ [3]

   J = { ☐ }

   Number of elements = ☐ [1]
   N (J) = ☐ [1]

10. Discuss with children:

    a. How they arrived at the number of each set. [Counting]

    b. Counting as the assignment of names to successively larger quantities.

    c. The Set of counting numbers { 1, 2, 3, . . .}

    d. That each successive counting number is one more than the number before it.

11. Children write in set notation:

    First five counting numbers
    Even numbers between 2 - 10
    Even numbers between 25 - 43
    Odd numbers between 2 - 10
    Odd numbers between 25 - 43

12. Discuss the endless set of counting numbers.

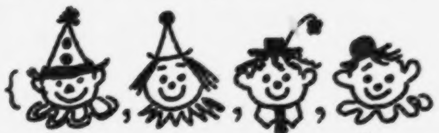
    Children note that no matter how far they count there are still more counting numbers.

    Tell children we use three dots (. . .) meaning "and so forth" to indicate such a continuation.

13. Review notation for the number of a set.

    [N (A)]

14. Present the following for children to complete:

A = {  } ---- Number of elements in A = ☐

B = { 8, 3, 2, 1, 6, 4 } ------- Number of elements in B = ☐

C = {Mary, Bob, John} --------- N (C) = ☐

D = { ✳ , Δ, O } --------- N (D) = ☐

E = { ✳ , O, Δ } --------- N (E) = ☐

They note that Sets E and D have the same elements and the same number. The numbers are equal; 3 = 3 Why?

[The quantity represented is the same; the cardinal number is the same; the idea of "three" is the same; the elements can be matched with none left over; etc.
The Sets contain the same elements.]

15. To obtain the (counting) number of a set, such as

$A = \{\square, \ \Delta, \ O, \ +\}$

Compare set A with the set of counting numbers $\{1, 2, 3, 4, 5, \ldots\}$ and set up the following one-to-one correspondence:

$$\begin{array}{cccc} \{\square, & \Delta, & O, & +\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \{1, & 2, & 3, & 4\} \end{array}$$

Have children notice our use of "counting" and that the last counting number needed is the number of the set.

PRACTICE AND / OR EVALUATION
SUGGESTED EXERCISES

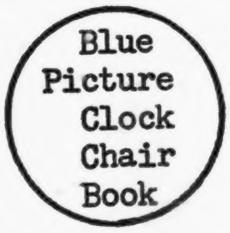
1. Write in set notation the set of counting numbers less than 15.

[{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 }]

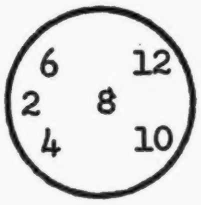
2. Show in set notation the set of all counting numbers greater than 30.

[{ 31, 32, 33, 34, . . . }]

3. Here are two collections of things.



A



B

Show in set notation the number of elements in each set.

[N (A) = 5
N (B) = 6]

4. Show that the counting number of a set you choose is 6.

[N { 0, x, □, ✳ , +, ÷ } = 6]

or

[A = { 0, x, □, ✳ , +, ÷ }
N (A) = 6]

5. Tell how you would find the number of elements in the set of girls in your class who are present today.

[By counting]

6. Write two different ways in which the set of girls may be described.

[By listing
By description]

7. Show two sets using set notation.
Compare these to show one-to-one correspondence.

SETS; NUMBER; NUMERATION

UNIT 3 - SET OF WHOLE NUMBERS: CONCEPT OF NUMBER AND NUMERAL; READING AND WRITING NUMERALS

NOTE TO TEACHER

Number and Numeral

To be able to distinguish between a System of Numbers and a System of Numeration, the difference in meaning between the terms number and numeral should be reviewed.

Number is the property of a set that denotes the idea of "How Many"; of quantity. This idea exists in the mind only.

A numeral is but the symbol or a name for a number. Three, Trois, Tres, 3, III are all symbols for the idea of the same number. Punched holes are symbols used with computers to convey the idea of number or to transmit information about number. A numeral is any symbol, agreed upon, for communicating the idea of number.

Children learn about the idea of quantity - the number of objects - before they read or write the symbols or numerals for representing the idea.

The symbol is not the number just as writing "blue" on the chalkboard is not the actual color blue. Symbols are simply ways of representing ideas of number.

Any number may be represented in many ways.

There are times when distinguishing between the words, "Number" and "Numeral" becomes cumbersome so that often we use the word

"Number" when we mean "Numeral". However, it is wise to try to use precise mathematical language whenever convenient.

The Set of Whole Numbers

The set of counting numbers, sometimes called the set of natural numbers is part of the set of whole numbers. The set of counting numbers can be symbolized in set notation as

(1, 2, 3, 4, 5 . . .)

Zero is not a counting number. The set of whole numbers includes zero which is the cardinal number of the empty set.

The set of whole numbers is recorded in set notation as

(0, 1, 2, 3 . . .)

Note that the set of whole numbers, unlike the set of counting numbers, has as its smallest number, zero. Like the set of counting numbers, the set of whole numbers has no greatest number.

Both sets are infinite (have an infinite number of elements).

TEACHING SUGGESTIONS

Objective: To help children understand:
  Meaning of number, numeral
  Zero as a whole number
  There is no greatest number

To reinforce:
  Counting
  Reading and writing numerals

Procedures

Concept of Number and Numeral

1. Children name other children in the classroom. Teacher lists the names,

then erases them. Discussion helps children realize that the children have not been erased.

Only the symbols or names for the children have been erased.

2. Use the same procedure for:
   Symbols on a musical scale and the notes they represent.
   Names for colors and the colors they represent, etc.

3. Display a set of six objects.
   Children identify the number of members of that set.
   Teacher records that number as: six, 6, VI, 𝍸 |
   Question the children:
   What is the same about these recordings?
   What is different about them?
   If we erase these symbols, have we erased the set of objects?

4. Tell children that the idea of "How Many" in the set is a number.
   The symbol is a numeral.

5. Teacher displays sets of various sizes.
   Children identify the number of each set.
   Children write the numeral for each number.

6. Discuss:
   Numbers are ideas which cannot be seen, written, erased.
   Numerals which stand for numbers can be seen, written, erased.

7. Write the following on the chalkboard and discuss the distinction between symbols and objects, number and numeral.

   A tiger - What do you see here?
   9, 5 - Which is larger?

The Set of Whole Numbers

1. Use a number line labelled from 0 to 20. Discuss:
   What is the next larger number after 5; after 3; after 15?
   What number comes just **before** 5; before 3; before 21?
   What is the next whole number after 1?
   Is there any number that comes just before 1?

2. Teacher writes numerals on the board: 16, 10, 8, 13, 0
   Ask children:
   What is the greatest number shown?
   What is the least number shown?

3. Tell children that the numbers under discussion are called whole numbers; that the set of whole numbers includes zero.

4. Teacher writes numerals on the board: 17, 38, 45, 0, 4
Direct children to rewrite them in order from least to greatest.
Ask children to continue to write numbers in order after the greatest number shown.

   Ask them what they think the largest number would be, were they to continue.

   Elicit from them that there is no largest number. Why?

5. Encourage children to tell what the set of whole numbers is.
   [The set of whole numbers is the set of numbers whose least number is zero, and which has no greatest number.
   Children will express this idea in their own words.]

6. Show children the following, {0, 1, 2, 3, 4 . . .}
   Discuss the use of the three dots.

PRACTICE and / or EVALUATION

SUGGESTED EXERCISES

1. List the set of the first 10 whole numbers.
   [{0, 1, 2, 3, 4, 5, 6, 7, 8, 9}]

2. List the set of whole numbers greater than 51.
   [{52, 53, 54 . . .}]

3. What is the greatest whole number in the set of whole numbers?
   [There is no greatest whole number]

4. List the elements of the following:
   The set of whole numbers between 100 and 101.
   [{ }]
   The set of whole numbers less than 5 and greater than 3.
   [{ 4 }]

5. List a set containing five elements each representing the number 9. Answers may vary.

$$[\ \{\ \mathrm{IX},\ 8+1,\ 4\tfrac{1}{2}+4\tfrac{1}{2},\ 3\times 3,\ 27\div 3\ \}\ ]$$

6. Children write the numerals for the following:

Five hundred seven
Twelve hundred twenty-five
Three thousand forty

7. Children write the following in words.

706 1342 3029 2500

8. Dictate numbers such as the following:

3286 41 1870 1028
605 2300 3 4009

Note: Number symbols such as 5280 may be read as 52 hundred eighty as well as 5 thousand 2 hundred eighty.

9. Children supply the missing numerals, counting by tens:
For example - 134, 144, 154

268, ____, ____ 1380, ____, ____
990, ____, ____ 3507, ____, ____

10. Children write the missing numerals in each sequence:

1970, 1980, 1990, ____, ____, 2020
1530, 1520, 1510, ____, ____, 1480
____, ____, 1000, 1001, 1002, ____
3500, 3600, 3700, ____, 3900, ____
____, ____, 2000, 2100, 2200, ____

11. Children count forward in each of the following series, limit determined by teacher.

Interval: Numbers through 9

47, 51, 55 . . .
69, 77, 85 . . .
178, 184, 190 . . .
305, 314, 323 . . .
577, 586, 595 . . .

Interval: 9 - 99

30, 60, 90 . . .
150, 200, 250 . . .
60, 120, 180 . . .
5049, 5059, 5069 . . .
3028, 3048, 3068 . . .

21, 42, 63 . . .
17, 34, 68 . . .
28, 56, 84 . . .
62, 124, 186 . . .
215, 230, 245 . . .

Interval: 99 - 999

300, 600, 900 . . .
600, 1200, 1800 . . .
700, 1400, 2100 . . .
2368, 2468, 2568 . . .
6072, 6272, 6472 . . .

250, 500, 750 . . .
130, 260, 390 . . .
402, 804, 1206 . . .
325, 650, 975 . . .
215, 430, 645 . . .

Interval: 1000, 2000, 4000, etc.

1000, 3000, 5000 . . .
1300, 2300, 3300 . . .
2529, 3529, 4529 . . .
1480, 3480, 5480 . . .
1461, 3461, 5461 . . .

12. Children count backward in each of the following series:

Interval: Numbers through 9

67, 63, 59 . . .
219, 211, 203 . . .
648, 643, 638 . . .
351, 342, 333 . . .
429, 422, 415 . . .

Interval: 100, 300, 700, etc.

1800, 1500, 1200 . . .
2428, 2328, 2228 . . .
3500, 2800, 2100 . . .
5472, 5272, 5072 . . .
3925, 3725, 3525 . . .

Interval: 30, 40, 50, etc.

180, 150, 120 . . .
350, 300, 250 . . .
300, 240, 180 . . .
5089, 5079, 5069 . . .
2098, 2078, 2058 . . .

Interval: 1000, 2000, etc.

9000, 8000, 7000 . . .
5300, 4300, 3300 . . .
7680, 6680, 5680 . . .
8320, 6320, 4320 . . .
6529, 5529, 4529 . . .

GEOMETRY AND MEASUREMENT

UNIT 4 - GEOMETRY: SETS OF POINTS; CURVES; NUMBER LINE

NOTE TO TEACHER

Points

We continue our exploration of Sets with their extension to Sets of Points.

A point is an idea.
A point cannot be seen or felt. It cannot be measured.

A point may be represented as a dot, as the end of a sharply pointed object, as the location where two walls and a ceiling meet, etc.

A point can be represented as a fixed location which does not move.

If the dot on the paper is erased, or the paper moved, the point still exists, and would have to be described in some other way, perhaps by a set of directions.

Space can be defined as the "set of all points".

Curves

Think of the idea of a path between two points in space.

All paths are sets of points in space and are called curves whether the paths are "straight" or not. The mathematical meaning of curve is different from the common meaning of curve.

A string stretched between two points, the representation on a map of the road between two cities are both representations of curves.

Curves include straight lines.

Line Segment

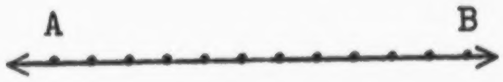
A line segment is an idea. It is a set of points that may be represented by a special curve drawn on paper connecting two dots. When we say "line segment" we mean "straight line segment".

The line segment is the shortest curve between two points. When represented it includes the end points of the line segment.

The symbol for a line segment is $\overline{AB}$. Points A and B are the end points of the line segment $\overline{AB}$,

Lines

A line can be thought of as the extension of a line segment in both directions.

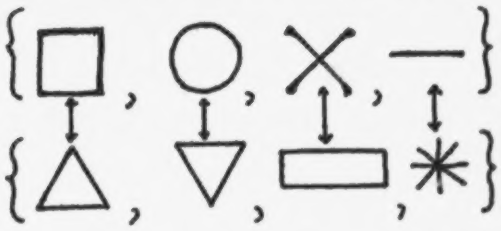


The symbol for the line above is: $\overleftrightarrow{AB}$
Notice how the two arrows indicate extension in both directions.

Number Lines: Sets of Points on a Line

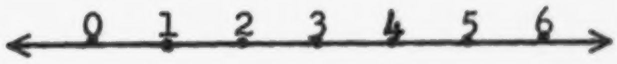
Because every point on a line has a position and because there is a distance between every two points on a line which may be compared with the distance between any two other points on that line, the number line is an invaluable device for helping children as they deal with operations on numbers and to see the one-to-one correspondence between some points on the line and the numbers under consideration.

We have previously discussed the association of a number with a set of things. For example: we associated "4" with any set whose members can be put into a one-to-one correspondence with any other set containing 4 elements, e.g.,



Both of these sets have a common number property. "4" is the counting number that tells how many elements in each of these sets.

"4" may also be associated with a specific point on the number line, thus:



There is on this number line, one and only one point corresponding to any one whole number.

The numbers are ordered. That is, they can be arranged in a sequence. Any number to the right of a number on the number line is greater than any number to the left of that number.

TEACHING SUGGESTIONS

Objective: To introduce geometric concepts of points and lines.
To introduce properties of lines.
To reinforce the number line.

Procedure:

1. Ask each child to touch a spot on his desk. Children locate the point.

   [6 inches from the right edge, 3 inches from the bottom edge.]

2. Teacher mentions a location that is not precise. For example: a spot to the right of the door and above the floor.

   Teacher asks children:

   Do you know exactly the spot meant? [No] Why not?
   How can you find out exactly the spot to which I am referring?
   [Answers will vary]

Teacher directs children to suggest ways of describing the spot so that it is exactly located.

3. Tell children that any precise location in space is called a point.

4. Teacher holds a sheet of paper against the chalkboard. Put a finger lightly on a point on the paper.Slide the paper away still keeping the finger at the same location.

Question children:

What did I do to the paper?	[Took it away]
Where is my finger?	[In the same place]
Did the point go away?	[No]

5. Children mark a dot on paper.
They erase the dot.
Elicit from children that the point is still in the same location even though the dot was erased.

6. Tell children that points, like numbers, are mathematical ideas. The dot is just a representation, symbol or model of the point, just as the numeral is a symbol for a number.

7. Discuss: How small is a mathematical point? ; a dot? ; etc.

8. Ask one child to hold the end of a string. Bring about the understanding that the end of the string is a point. Ask another child to hold the other end of the string. Establish this end as another endpoint.

Elicit from the children that the string makes a path from one point to another point.

Have the children stand so that the string is not taut.

Discuss idea of a "curved line".

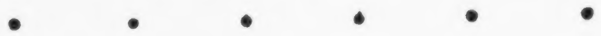
They stretch the string taut.
Compare curved path with straight path.

[straight path yields the shortest distance between 2 points]

Children drop the string. Establish, through discussion, that the path the string traversed is still there, even though the string is not.

9. Children mark a dot on the paper. They mark a series of dots

in line with the first dot.



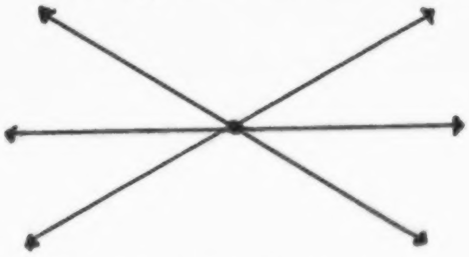
They place dots in between the dots already on the paper.

They continue to place dots between the existing dots until they are dense.

They draw a line through the dots.
They note a line is a set of points.

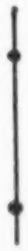
10. Children mark a dot on paper.
They draw a straight line through the dot.
They continue to draw lines through that point.

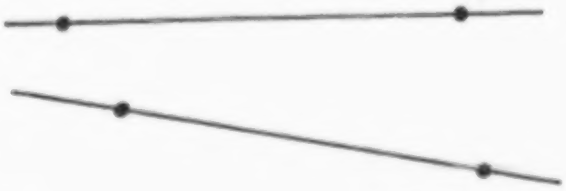
Children note that any number of lines can be drawn through one point.



11. Direct children to mark a dot on a paper. They mark a second dot a distance away in any direction.

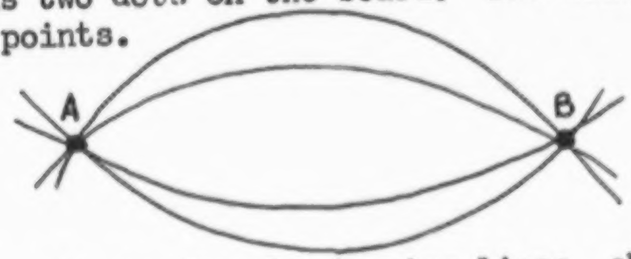
They draw straight lines passing through the points.



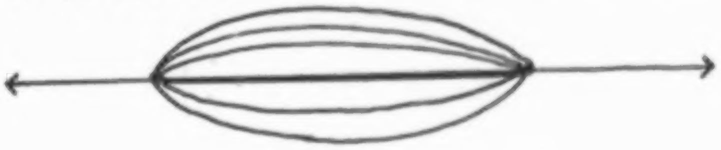


They see that one and only one straight line can be drawn through two points in one plane.

12. Teacher marks two dots on the board. She draws many curved lines between the points.

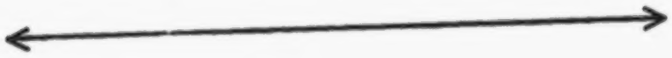


Using strings first, then by drawing lines, children experiment to find the shortest curve between points A and B.



Discuss these curves until children see that all lines are curves but that the curve that is the shortest distance between two points is a straight line segment.

13. Draw a line.



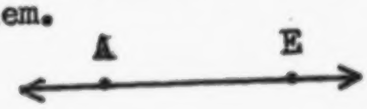
Ask children:

Why are the arrow heads there?
What are the end points of the line shown?

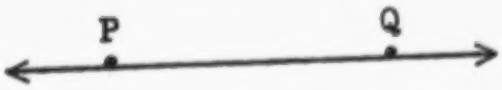
[There are no end points on a line.]

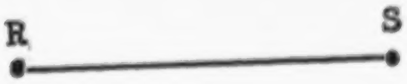
Tell children that a line is named by any two points on the line with the symbol " ↔ " above them.

$\overleftrightarrow{AE}$ names the line



14. Present the following:





Discuss the line and line segment until children understand that a line segment is part of a line; that a line segment has specified end points.

In the diagram above "R" and "S" are end points of that line segment.

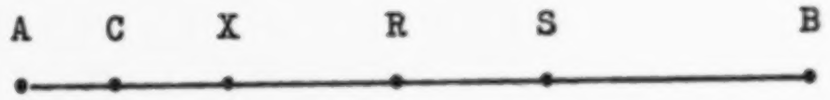
Tell children that the symbol for a line segment is " — " .

The line segment above is symbolized as "$\overline{RS}$"

15. Children draw, identify and symbolize lines and line segments.

16. Direct children to place a dot on a paper. They label it "A" . They place another dot and label it "B" . Children draw a line segment from A to B.

Ask children to mark off a series of points along the line segment. They label each point.



Discuss points on a line.

Can you always insert another point?
How many points do you think are on a line segment?
[An undetermined number.]

17. Children are familiar with the number line.

Emphasize through discussion that:

We are assigning whole numbers to certain points on the line.

Each number can be considered as a name for that point.

Zero is the least whole number.

Numbers to the right of any number are larger than that number.

Numbers to the left of any number are smaller than that number.

PRACTICE AND / OR EVALUATION
SUGGESTED EXERCISES

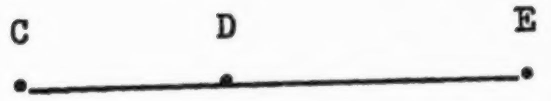
1. Which of the following is the best model of a point? Why?

● · •

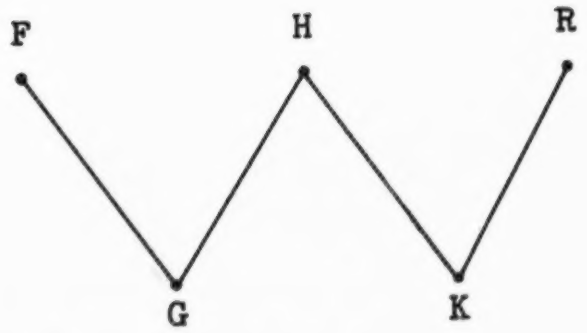
2. Which of the following are the best answers? Why?

A point is a dot.
A point is an exact location in space.
A point is the end of a nail.
A point is an idea.

3. Name all the line segments represented in each figure below.

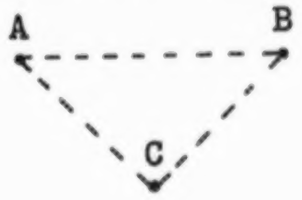


[$\overline{CD}$
$\overline{DE}$
$\overline{CE}$]



[$\overline{FG}$ $\overline{GH}$ $\overline{HK}$ $\overline{KR}$]

4. Draw all possible line segments to connect the given points.
Name all the line segments.



[$\overline{AB}$ $\overline{BC}$ $\overline{CA}$]

5. Note points "P" and "R" below.
How many line segments can be drawn with end points "P" and "R" ?

* 6. Exploratory Exercises

Draw sketches to show:

How many points can two straight lines always have in common?

Will 2 straight lines always have at least one point in common?

[Remember the definition of a line.]

GEOMETRY AND MEASUREMENT

UNIT 5 - MEASUREMENT: TEMPERATURE

NOTE TO TEACHER

The derivation of the word Geometry comes from "geo", the earth and "metry," measurement. Formerly Geometry was concerned mainly with measurement of the earth.

Measurement is the process of assigning numbers to physical objects or physical quantities or to their mathematical abstraction, line segments etc.

All measurement involves a comparison with a unit of measure. The number of units in the object being measured is called the measure of this object.

All physical measurement is approximate.

When counting objects, the answer to the question, "How many are there?" can be exact.

However, in the measurement of distance, capacity, weight, time, etc., absolute accuracy is impossible. While we are saying, "It is 6 o'clock" the second hand of the clock has already moved on, and it is after 6 o'clock. When measuring with a standard ruler the width of the markings on the ruler and the degree to which the lead of the pencil is sharpened, contribute to variations.

Measurement concepts will be developed by using instruments and units of measure in experience situations. They include: schedules for railroad, bus and airplane travel; time tables for radio and television programs; records of changes in tides and temperature; time of sunrise and sunsets, etc.

Before using standard units of measure, children should be given many opportunities to use non-standard units such as hand, span, foot, unmarked containers.

Children should have many opportunities to estimate length, weight, height, etc. They should check their estimates with measurements made.

Standard instruments (scales, rulers, calendars) for measuring weight, length, time, etc. should be available.

Give children many opportunities to determine the appropriate instrument that can be used in a given situation. For example: in testing vision in the classroom, the distance from the eye chart is 20 feet. What instrument should we use to measure the distance? Would you use a 6 inch ruler or a yardstick? Why? etc.

Provide ample practice for expressing estimation of quantity in the most suitable unit of measure. The length of a room can be estimated as 18 feet; a piece of paper as 12 inches long.

Indirect Measurement

Some objects cannot be measured directly; their measurements are arrived at by indirect methods. We may call these "indirect measurements". For example:

> The height of a very tall building, a tree or an inaccessible object such as a mountain, is arrived at indirectly. We measure directly certain lengths and angles, and then perform computations on the (approximate) measures obtained.

Temperature

It is not possible to measure temperature directly. To "measure" temperature, we make use of a property of heat. When a substance gets warmer, it usually expands. The instrument used to measure temperature is the thermometer in which a column of liquid expands or contracts in relation to the amount of heat to which it is exposed. We measure the length of the column of liquid to estimate the temperature. The unit of measure on the thermometer is the degree.

The thermometer can be considered a number line. Two scales, the Fahrenheit and the Centegrade, can be introduced. Tabular and graphical comparisons can be made. An important use of these scales is the opportunity to work with positive and negative numbers.

TEACHING SUGGESTIONS

Objective: Continued development of concepts of temperature.

Procedures

1. Have several thermometers available for the children.
   Children hold their hands around the bulb of the thermometer to discover that:

   Mercury is a liquid that easily expands and contracts.
   The measure of the length of the column of mercury indicates temperature.

2. Discuss the:

   Scale on the thermometer as part of a number line.
   Markings on the thermometer as end points of the segments.
   Variations of length of intervals on different thermometers (Scale).
   Measurement and measurement of temperature as an approximate value and as an indirect measure.

3. Discuss the:

   Unit of measure for temperature as a degree.
   Symbol for the unit, " ° "
   Meaning of zero degrees on the thermometer
   Boiling point and freezing point on the Fahrenheit Scale.
   Variations of boiling point and freezing points of various substances.
   Include milk, alcohol, mercury, water, etc.

   Boiling Point of Water = 212° F
   Freezing Point of Water = 32° F etc.

4. Children read and interpret the readings on various types of thermometers. They read temperatures below zero.

5. Use experience charts to compare outdoor and indoor temperature, daily temperature, etc.

	Temp. at 10 A.M.	Temp. at 2 P.M.
Mon.	60°	68°
Tues.	48°	57°
Wed.		
Thurs.		
Fri.		

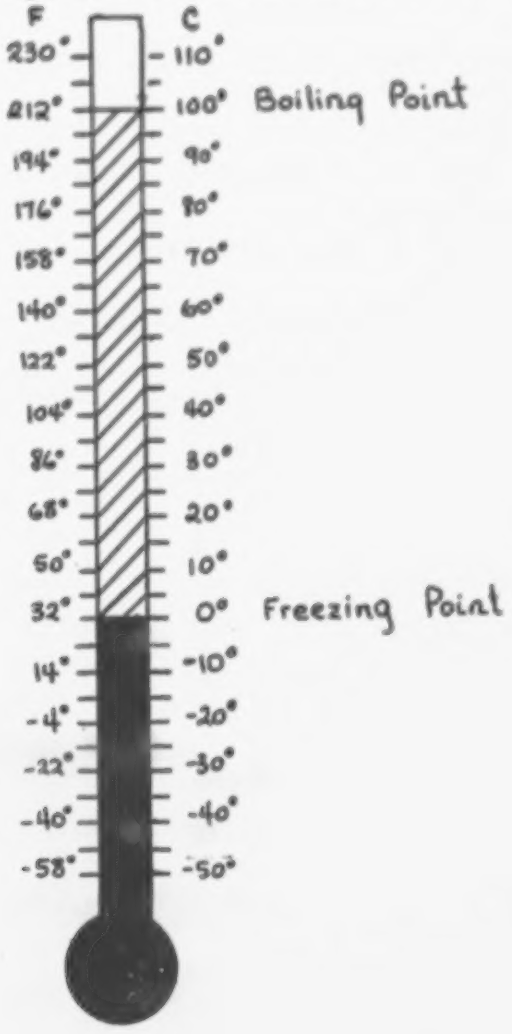
	Outdoor Temp.	Indoor Temp.
Mon.	60°	72°
Tues.	48°	72°
Wed.		
Thurs.		
Fri.		

✱ 6. Discuss idea of Centigrade Scale **(Optional)**

0° C = 32° F

100° C =212° F

Centigrade is used in European countries and in Science.


F
C
230°
110°
212°
100°
Boiling Point
194°
90°
176°
80°
158°
70°
140°
60°
122°
50°
104°
40°
86°
30°
68°
20°
50°
10°
32°
0°
Freezing Point
14°
-10°
-4°
-20°
-22°
-30°
-40°
-40°
-58°
-50°

EVALUATION and / or Practice
SUGGESTED EXERCISES

1. Which is warmer, - 10° or - 20° ? [-10°]

2. If the temperature at 8 o'clock is 2° below zero and at noon it is 16° above zero, how much warmer has it gotten?
[18 degrees warmer]

3. The temperature this morning was 10°. It has gotten 15° colder. What is the temperature now? [-5°]

4. Use variously scaled thermometers to answer the following:

   How many spaces are there on the scale from the 60° to the 70° mark, etc.

   How many degrees does each space represent?
   [Answers will vary depending upon the thermometer used]

5. Match the following temperatures with the appropriate item in the column at the right.

350°	Ice-skating Weather
32°	Boiling Water
10°	Swimming Weather
212°	Temperature at which water freezes
95°	Oven Temperature

6. The temperature changed from 5° to 2° below zero last night. This afternoon it went from 8° to 14° . Which was the greater change?
[5° to 2° below zero]

7. Relate to science experiment.

   Materials needed: Colored water, alcohol, mercury, long glass tubes.

   Pour a small amount of a liquid into a tube.
   Children cover the base of the tube with their hands and note how the liquid expands and rises in the tube as it gets warmer.

   They note which liquid reacts most quickly.

   Children construct a scale for each tube to note variations in temperature.

8. Additional exercises may be found in textbooks.

OPERATIONS

UNIT 6 - SET OF WHOLE NUMBERS: UNION OF SETS; ADDITION

NOTE TO TEACHER

Two sets may be combined or joined to form a new set.

The new set is called the union of the two sets. The term "union" is applied both to the operation and to the resulting set.

Understanding of the union of two sets is basic to the understanding of addition of whole numbers.

Sets having no elements in common are called Disjoint Sets. For example,

A = { Susan, Mary, Judy }
B = { Ken, Frank, Paul}

are disjoint sets.

The union of Set A and Set B is a new Set, C, where

C = {Susan, Mary, Judy, Ken, Frank, Paul}

The symbol for union is " U ".

Here C = A U B

If X is the set of boys in the classroom and if Y is the set of girls in the classroom, we form a union of Sets X and Y to arrive at the Set of children in the room, Set Z.

We record this as: Z = X U Y

Union of Sets and Addition of Numbers

The term "Union" applies to sets and the term "Addition" applies to numbers.

For example,

$$\text{Let } A = \{ a, b, c \}$$

$$B = \{ h, t, b \}$$

$$\text{Then } A \cup B = C \text{ where}$$

$$C = \{ a, b, c, h, t, b \}$$

$$\text{and } N(A) + N(B) = N(C)$$

$$3 + 3 = 6$$

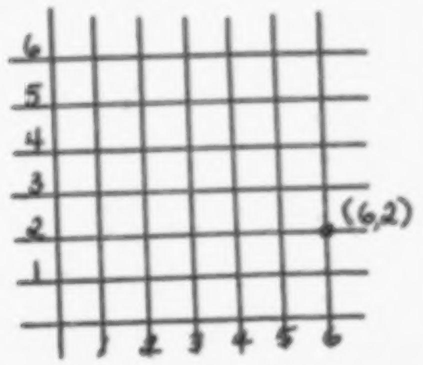
Addition is a binary operation in which an ordered pair of numbers is operated on to yield a third number called the sum. Each number of the ordered pair is called an addend.

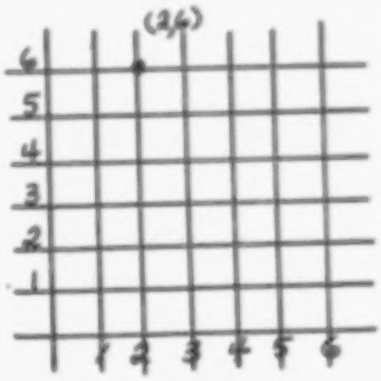
An ordered pair of numbers involves two considerations:

The numbers

The order in which they are considered.

For example: The ordered pair (6, 2) is not the same as the ordered pair (2, 6) although both involve the same pair of numbers and both yield the same sum. This can be shown on the quadrants below.





Definition of the Operation of Addition for Whole Numbers

If A and B are disjoint sets as above, we have seen that:

$$N(A) + N(B) = N(A \cup B) \quad \text{since} \quad C = A \cup B$$

This is the way we can define the operation of the addition of two whole numbers which we will call a and b:

If $N(A) = a$
and $N(B) = b$
Where A and B are disjoint sets
then $a + b$ is equal to $N(A \cup B)$

By applying this definition we can verify each of the following properties.

Properties of Addition: Set of Whole Numbers

Operations on numbers are subject to certain rules or properties.

Some properties of operation for addition are the:

Associative Property

When three or more numbers are to be added, the order in which they are grouped does not affect the sum.

For example:

4, 3, 2 may be grouped as
$(4 + 3) + 2$ or as
$4 + (3 + 2)$
and $(4 + 3) + 2 = 4 + (3 + 2)$

Commutative Property

The order in which two numbers are added does not affect the sum.

For example:
$3 + 4 = 4 + 3$

Identity Element for Addition is Zero

$6 + 0 = 6$ and $0 + 6 = 6$

Closure

The set of whole numbers is closed with respect to addition.

When whole numbers are added, the sum is always a whole number.

Concepts from Algebra

A mathematical sentence is any statement of equality or inequality involving numbers.

A mathematical sentence may be either open, true, false .

An open sentence contains one or more place holders.

$4 + \square = 9$	is an example of an open sentence
$4 + 5 = 9$	is an example of a true sentence
$\{5\}$	is the truth set for the open sentence $4 + \square = 9$

A given set or a set from which it would be reasonable to choose the number for the truth set is called the replacement set. For $4 + \square = 9$ the set of whole numbers would be considered the replacement set.

The symbols $\square$ or n or Δ, etc. are called placeholders or variables.

$8 > (n \times 2)$ is also an open sentence.

If we establish the replacement set as the set of whole numbers, then the truth set for $8 > (n \times 2)$ is $\{0, 1, 2, 3\}$

$4 + 5 = 8$ is an example of a false statement.

Basic Addition Facts

An addition fact is a statement about the two addends, each from 0 through 9 and their sum - a number from 0 through 18. Thus:

$8 + 4 = 12$ is an addition fact but
$12 + 4 = 16$ is not called an addition fact.

Teachers must aim for automatic response to basic facts by their children.

Automatic response is achieved through periodic drill.

Drill

Drill should be organized according to specific patterns which emphasize one type of relationship at a time.

A few minutes of each lesson should be devoted to drill.

Drill should be related to the major topic under consideration, wherever possible.

TEACHING SUGGESTIONS

Objectives:

To develop understanding of the union of sets.
To relate addition of numbers to union of sets.
To test and / or reinforce automatic response to addition facts.
To reinforce extensions of addition facts to higher decades.
To introduce concepts from algebra.

Procedures

Union of Sets

1. Display two sets of objects, for example, a ruler and a scissors; a crayon and a pencil

   Children note that the objects are different.

2. Children show these in set notation, as:

   Let A = { scissors, ruler }

   Let B = { pencil, crayon }

3. Ask children how to form a new set using the elements of A and B.

   [Put them together, join them, etc.]

4. Ask children:

   What are the elements of this new set which we will call Set C?

   Teacher records as child responds

   C = {scissors, ruler, crayon, pencil}

   Does Set C include all of the elements of Set A? of Set B?

5. Tell children that the symbol for joining sets is " ∪ " and that it is read as: "Union of".

   Record in set notation the action of joining the sets.

   {scissors, ruler} ∪ {crayon, pencil} = {scissors, ruler, pencil, crayon}
   A ∪ B = C

   Emphasize that the term "union" applies to sets.

6. Reinforce set notation:

   Braces; symbol for empty set; symbol for union; symbol for the number property of a set.

7. Present the problem:

   Alice, Joan and Mary went skating. At the rink they met Harry, Tom and Bob. The two groups skated together.

   Children show this in set notation .

   { Alice, Joan, Mary } ∪ { Harry, Tom, Bob }

   Children show the union of the two sets.

   Let A = {Alice, Joan, Mary}
   B = {Harry, Tom, Bob}
   C = {Alice, Joan, Mary, Harry, Tom, Bob}

   Then , A ∪ B = C

8. Use the problem above to reinforce number as a property of sets.

Children show, in set notation, the number of elements in Set A, Set B and Set C as:

N(A) = □ [3]
N(B) = □ [3]
N(C) = □ [6]

Relate Union of Sets to Addition of Numbers

Ask children (referring to the sets above)

What are the elements of Set A?

[Alice, Joan, Mary]

What is the number of Set A?

[N(A) = 3]

What are the elements of Set B?

[Harry, Tom, Bob]

What is the number of Set B?

[N(B) = 3]

What are the elements of Set C?

[Alice, Joan, Mary, Harry, Tom, Bob]

What is the number of Set C?

[N(C) = 6]

Show in set notation that the number of Set A added to the number of Set B , will give the number of Set C where C = A U B.

$$\left[\begin{array}{l} N(A) + N(B) = N(C) \\ N(A) + N(B) = N(A \cup B) \end{array}\right]$$

Show the above using only numerals.

[3 + 3 = 6]

Emphasize that:

The binary operation "Union" applies to sets. We combine sets.

The binary operation "Addition" applies to numbers. We add numbers.

Facts and Extensions

1. Test for automatic response to basic addition facts.

   Drill or reteach as indicated by results.

2. Test children's responses to the extension of addition facts to higher decades.

A

Sums in the Same Decade

17 + 2 = □
24 + 2 = □
46 + 2 = □

15 + 4 = □
26 + 4 = □
64 + 4 = □

B

Sums reaching the Next Decade

41 + 9 = □
14 + 6 = □
38 + 2 = □

17 + 3 = □
55 + 5 = □
66 + 4 = □

C

Sums in the Next Decade

57 + 5 = □
46 + 5 = □
14 + 8 = □

89 + 7 = □
18 + 3 = □
59 + 8 = □

D

Sums in the Hundreds

324 + 3 = □
436 + 4 = □
518 + 5 = □

Sums in the Thousands

- Children make the open sentences true.

1324 + 3 = □
2436 + 4 = □
1617 + 6 = □
1598 + 5 = □

Read as hundreds rather than as thousands to facilitate arriving at sums. For example:

1324 + 3 is read as: 13 hundred 24 plus 3

3. Drill or teach as indicated by results, emphasizing the following mathematical relationships.

Applying Commutative Property

2 + 9 thought of as 9 + 2
5 + 23 thought of as 23 + 5
6 + 184 thought of as 184 + 6

Deriving Near-Doubles from Doubles

From 8 + 8 we derive, 8+7, 8+9, 7+8, 9+8
From 25 + 25 we derive, 25 + 24, 25 + 26, 24 + 25, 26 + 25

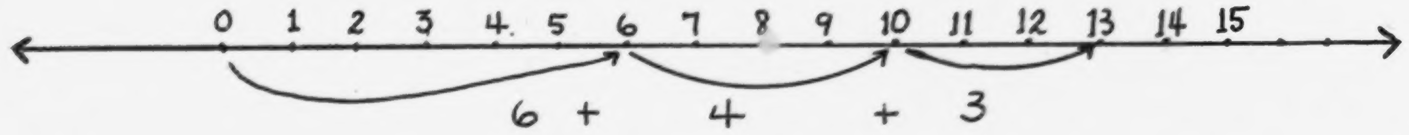
Applying Associative Property

9 + 6 thought through as
$9 + (1 + 5) = (9 + 1) + 5 = 10 + 5 = 15$

38 + 5 thought through as
$38 + (2 + 3) = (38 + 2) + 3 = 40 + 3 = 43$

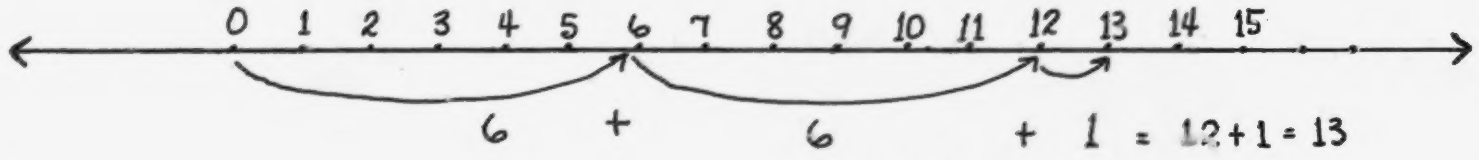
243 + 9 thought through as
$243 + (7 + 2) = (243 + 7) + 2 = 250 + 2 = 252$

4. Use the number line to reinforce addition facts and extension of facts to higher decades.



$6 + 7 = 6 + (4 + 3) = (6 + 4) + 3 = 10 + 3 = 13$

or



Extend number line for higher decades.

Concepts from Algebra

1. Tell children that:

   Symbols □, Δ, n, ◇, etc. in a mathematical sentence are called placeholders or variables.

   Mathematical statements containing variables are called open sentences.

2. Direct children:

Replace the variable to make a true statement

$$8 + \square = 17$$
$$15 + \square = 21$$
$$9 + \square = 14$$

Which of the statements below are true? Which are false? Which are open?

$$3 \times 7 = 21$$
$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
$$\square - 7 = 10$$
$$48 + 22 = 60$$
$$\frac{1}{2} - \frac{1}{4} = 1\frac{1}{4}$$

Why cannot $\square - 7 = 10$ be called a true statement ;A false statement?

Write an open sentence; A true sentence; A false statement?

3. Make this open sentence true: $236 + \square = 243$ [7]

Tell children that $\{7\}$ is called the truth set for the open sentence $236 + \square = 243$. Ask, "Why?"

Let children find the truth set for the following open sentences. In each case ask how they obtained the answer.

$6 + \square = 27$; $3 + (2 \times \square) = 11$; $2 + (3 \times \square) = 2$

4. Have children make the following open sentences true.

Commutative Property

$3 + 36 = 36 + n$
$7 + 148 = n + 7$
$9 + 2145 = 2145 + n$
etc.

Associative Property

$105 + 7 = 110 + n$
$1526 + 4 + \square = 1532$
$1397 + 8 = 1400 + \square$
$225 + 5 + n = 233$

Use of the terms "Commutative" and "Associative" would depend upon the maturity of the children.

5. Replacement Set

Given the set $\{7\frac{1}{2}, \frac{1}{2}, 3, 5, 6\}$ children choose the values for n to make the following statement true.

$8 > (n + 2)$ [$n = \frac{1}{2}$, or 3, or 5]

Tell children that the set of numbers from which one or more numbers can be chosen to replace a variable is called a replacement set.

Why is the name, "Replacement Set" given to this set?

Children note that more than one choice is possible for the truth set or solution set.

6. Children find the truth set for the following:

Open Sentence	Replacement Set
a. $4 < \square < 9$	The set of whole numbers
b. $\square + \Delta = 7$	$\{1, 2, 3 \ldots 10\}$
c. $37 + 5 > n$	$\{43, 41, 29, 50\}$

PRACTICE and / or EVALUATION
SUGGESTED EXERCISES

1. Using any or all of the following: 4, 6, 8

   Write two statements of inequality.

2. Tell what whole number n is, so that each mathematical sentence below, is true.

   $n + 50 = 50 + n$ [$\{0, 1, 2, 3 \ldots\}$ or any whole number]

   $13 + \square = 24$

   $9 + 8 < n$ [$\{18, 19, 20 \ldots\}$]

   $26 + 13 > n$ [$\{0, 1, 2, 3 \ldots 38\}$]

3. Additional exercises may be found in textbooks.

SETS; NUMBER; NUMERATION

UNIT 7 - SUBSETS

NOTE TO TEACHER

A set contained in another set is called a subset of the original set.

Each element of the subset is by definition an element of the given set.

For example: All of the children in a classroom may be thought of as a set of children.

The set of girls is a subset of the set of children, The set of boys is another subset of the class of children.

The set of even numbers is a subset of the set of whole numbers.

The set of odd numbers is a subset of the set of whole numbers.

The symbol for "is a subset of" is "$\subset$" .

If Set $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and
Set $B = \{2, 4, 6, 8\}$

Then Set B is a subset of Set A because each of the elements of Set B is also an element of Set A.

This is symbolized as: $B \subset A$ and is read as: Set B is a subset of Set A.

Understanding of subsets is important in arithmetic. For example: We can explain subtraction of two whole numbers by removing a subset from a given set and examining the number of the three sets involved.

TEACHING SUGGESTIONS

Objective: To help children understand the meaning of subset.

Procedures

1. The problem of rearranging the bookshelves is presented to the class. Three children, Mary, Alice and John are chosen.

2. Identify these children as elements of a set.

3. Children record this given set using set notation.

   [{Mary, Alice, John}]

4. Guide children to see that there are sets within a given set.

   Ask children:

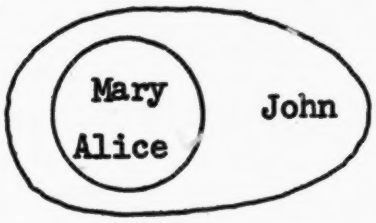
   Which children within this set can we use to help rearrange the books on the back shelves?

   [Answers may vary]

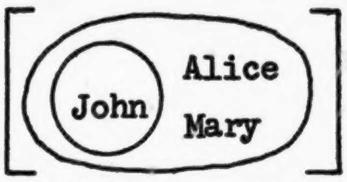
   How would you describe Mary and Alice in relation to the set?

   [Elements of the set]

   Use a diagram to show this.



   Show, using diagrams, other children from the given set who might be used to help.



   Record different sets of helpers, using set notation.

   [{Mary, John}] etc.

   Ask children:

   What can you tell about the relationship of the sets you have recorded to the given set.

[Children helping are in the set of children you chose;
Mary and Alice are part of the names you put on the board;
etc.]

5. Tell children that sets that are part of a given set are called subsets of that set.

6. Ask children whether we might use Alice, Mary and John to help.

[Yes]

Tell children that any set may also be considered a subset of itself.

7. Ask children what notation they would use to show that we do not wish to use any of the children of the given set.

[$\{ \}$]

Ask children whether there is any member in the empty set that is not an element of the given set.

Tell children that because there is no element in the empty set that is not in the given set the empty set is a subset of every set.

8. Tell children that the symbol for subset is " $\subset$ " .

For example: $\{ \text{Alice} \} \subset \{ \text{Alice, John, Mary} \}$ is read as: The set that contains Alice is a subset of the set whose elements are Alice, John, Mary.

We may also record this in another way.

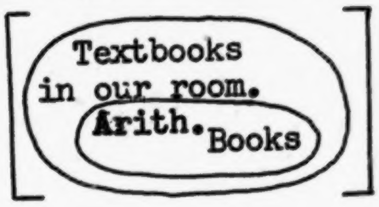
$$D = \{ \text{Alice, John, Mary} \}$$

$$F = \{ \text{Alice} \}$$

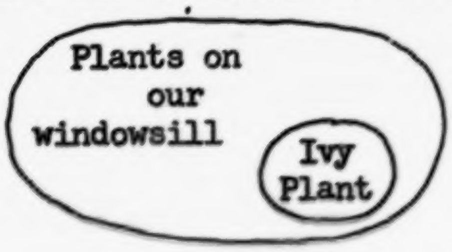
Then we may record the same idea as: $F \subset D$ read as: "Set F is a subset of Set D".

PRACTICE EXERCISES

1. Use a diagram to show that the set of Arithmetic books is a subset of the set of all the textbooks in our room.

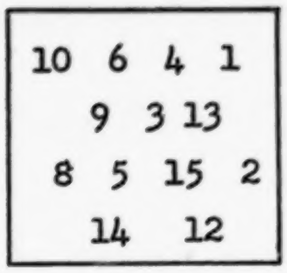


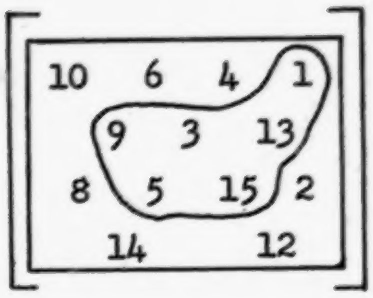
2. Use set notation to describe the diagram below.


Plants on our windowsill
Ivy Plant

[{Ivy plant} ⊂ {Plants on our windowsill}]

3. Consider the diagram below. Encircle the subset of odd numbers.


10 6 4 1
9 3 13
8 5 15 2
14 12


10 6 4 1
9 3 13
8 5 15 2
14 12

4. Use data on the chart to give information required about {John, Tom, Don, Alan, Bill}

Name	Color of Tie	Color of Shirt
John	Blue	White
Don	Red	Grey
Alan	Grey	Blue
Bill	Blue	White
Tom	Grey	Grey

a. Show in set notation that the boys whose ties are blue is a subset of the given set.

[{John, Bill} ⊂ {John, Don, Alan, Bill, Tom}]

b. Show in set notation that the boy wearing a red tie is a subset of the given set.

[{Don} ⊂ {John, Don, Alan, Bill, Tom}]

c. If Frank also is wearing a red tie is the following statement true? Explain.

{Don, Frank} ⊂ {John, Don, Alan, Bill, Tom}

5. List in set notation all the subsets of the set of rivers around New York City.

{Hudson, Harlem, East}
{Hudson} ; {Hudson, East} ; {Hudson, Harlem}
{Harlem} ; {Harlem, East}
{ East } ; { }

*6. How many subsets are there in a set that contains 3 elements? (Optional)

*7. Distinguish between: (Optional)

$$\text{Alice} \in \{\text{Alice, John}\}$$

and

$$\{\text{Alice}\} \subset \{\text{Alice, John}\}$$

*8. How many subsets are there of a set that contains 4 elements? (Optional)

OPERATIONS

UNIT 8 - SET OF WHOLE NUMBERS: SUBTRACTION - RELATED TO COMPLEMENT OF A SET; MEANINGS; PROPERTIES; FACTS; EXTENSIONS TO HIGHER DECADES

NOTE TO TEACHER

Although children have dealt with subtraction in earlier grades a recapitulation of subtraction in connection with sets may help them to solve verbal problems.

An interpretation of subtraction in the set of whole numbers depends upon:

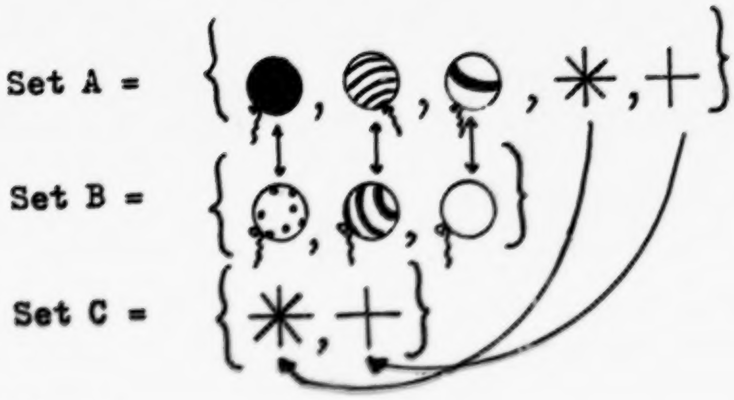
1. The meaning of addition in the set of whole numbers in terms of the union of disjoint sets.
2. The understanding that subtracting a number is the inverse operation of adding that number.

Interpretation of Subtraction in Terms of One-to-One Correspondence; Related to Finding the Difference in Subtraction of Numbers

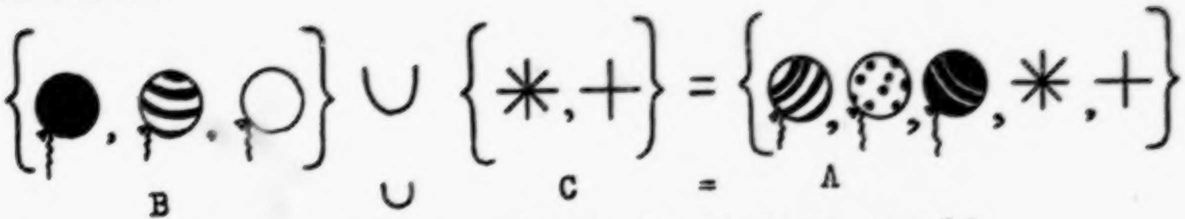
Illustrative Example

Choose a set with 5 elements, Set A and a set with 3 different elements, Set B.

Find another Set C such that when the union of Set B and Set C is formed, the elements of this union can be placed in one-to-one correspondence with the elements of Set A.

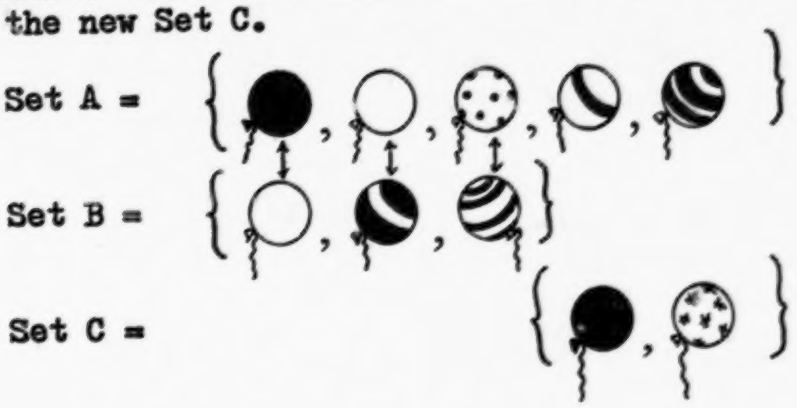


Therefore:



The elements of the union of sets B and C may now be placed in one-to-one correspondence with the elements of Set A.

When the elements of Set A are matched with the elements of Set B the elements that are not matched form the new Set C.



Relating Matching of Sets to Subtraction of Numbers

Referring to the conditions stated above, we find that if we know the number of elements in Set A (5) and the number of elements in Set B (3), we can find the number of elements in Set C (2).

We may think in terms of addition:
What number added to 3 will result in 5?
This may be stated as: $3 + \square = 5$
We must find a second addend which when added to 3 will give the sum 5.

Addition is an operation on two numbers, to find a third number called the sum. The two numbers are called addends.

Addend + Addend = Sum
3 + 2 = □

Subtraction then is the operation of finding a missing addend when the sum and the other addend are known.

Addend + ☐ = Sum
3 + □ = 5

In terms of subtraction this is finding the difference.

Interpretation In Terms of Subsets (Complement of a Set); Related to Finding "How Many Are Left" In Subtraction of Numbers

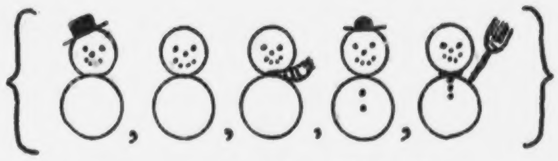
Illustrative Example

Choose a Set A with 5 elements from which we wish to remove a subset Set B of 3 elements. The subset that is left can be called the remainder set, which we will call Set C.

Set C is said to be a complement of Set B relative to Set A.

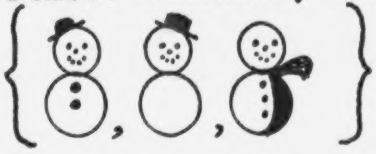
For example:

Set A =



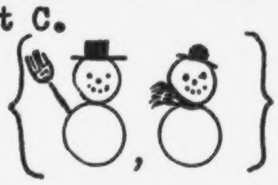
From Set A we remove a subset, Set B

Set B =



We are left with the elements of another subset which we will call Set C.

Set C =

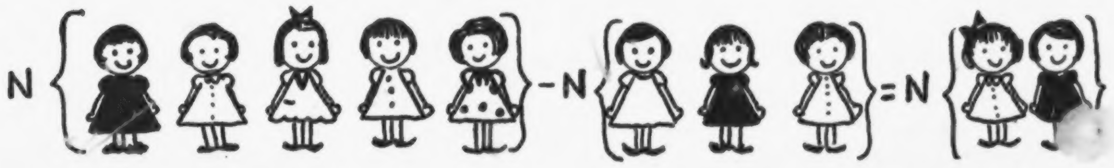


Set C is the remainder set.
Set C is the complement of Set B relative to Set A.

Should we wish to remove Subset C from Set A, then Subset B would be the remainder set or the complement of Set C relative to Set A.

Relating Removing of a Subset to Subtraction of Numbers

When we wish to remove a number of elements from a set the operation of subtraction is involved:



N (A) - N (B) = N (C)

N (A) = 5   N (B) = 3

5 - 3 = 2

Subtraction is a binary operation on an ordered pair of numbers.

A binary operation means that only two numbers may be operated on at any one time to produce a third number.

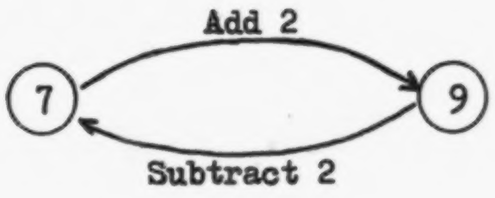
Finding "The Difference" and finding "How Many Are Left" situations are both solved by the operation of subtraction.

Some Properties of Subtraction

Inverse Operation

An action often has associated with it an inverse action. We open a door, we close a door; we put on shoes, we take off shoes.

Mathematical operations also have inverse operations. For example: After we add 2 to 7 to arrive at 9, to get back to 7 we subtract 2 from 9.



This may be thought of as a "doing" and an "undoing."

"Subtracting a number" is the inverse operation of "adding that number."

Addition. . . . . $7 + \underline{2} = 9$

Related Subtraction . . . $9 - \underline{2} = 7$

Commutative Property

Subtraction is not commutative

$4 - 3 = 1$ but
$3 - 4 \neq 1$

Associative Property

Subtraction is not Associative

$(6 - 4) - 2 = 0$ but
$6 - (4 - 2) \neq 0$

Property of Closure

The set of whole numbers is not closed with respect to subtraction. When whole numbers are subtracted the result is not always a whole number. For example: $6 - 8$ is not within the set of whole numbers.

Role of Zero in Subtraction

Zero subtracted from any number results in that same number.

$$6 - 0 = 6$$

However, a whole number subtracted from 0 does not result in a whole number or in that same number.

$$0 - 6 \neq 0$$
$$0 - 6 \neq 6$$

(Compare with addition where $6 + 0 = 6$ and $0 + 6 = 6$).

Basic Subtraction Facts

A basic subtraction fact involves three numbers, at least two of which are selected from 0 through 9. For example:

$15 - 7 = 8$ is a subtraction fact

$25 - 7 = 18$ is not considered a basic subtraction fact. It is an extension of a basic subtraction fact.

Drill and Testing

Subtraction facts should be presented concurrently with their related addition facts.

$$8 + 7 = 15; \quad 15 - \square = 8$$

TEACHING SUGGESTIONS

Objectives: To relate the operation of subtraction to relative complements of sets. (The "take away" aspect of subtraction).

To relate the operation of subtraction to ideas of Union of Sets and One-to-One Correspondence. (Comparison Difference aspect of subtraction).

To apply properties of Subtraction to drill and/or reteaching of facts and extensions of facts to higher decades.

Procedures:

Finding a Remainder Set - Relative Complement of a Set

1. Direct 5 children to come to the front of the room. Mary, Jane, Eva, Al, Tony. This is the set of children who will arrange books. Al and Tony are called to the office.

   Ask children to:

   Record set of children who were to help arrange books.
   Call this Set A .
   A = {Mary, Jane, Eva, Al, Tony}

   Record subset of children called out of room.
   Call this Set B.
   B = {Al, Tony}

   Record the set left to help with books.
   Call this Set C.
   C = {Mary, Jane, Eva}
   Why can we not call it Set A?

   Tell children Set C is called the remainder set.

2. Show a set of objects on desk.
   Ask children to come and remove a subset.
   Ask children to describe the remainder set.

3. Display a set of objects, e.g. a disc, a book, a pencil, a plant, a jar of paint. Call it Set A.

   Ask children to record this set showing the number of elements. [N(A) = 5]

   Ask children to remove a subset whose number is 2 from Set A.
   Call the subset, Set B.
   Ask children to record this subset showing the number of elements. [N(B) = 2]

   Call the remainder set, Set C.
   Ask children to record the number of elements in the remainder set. [N(C) = 3]

4. Ask children to write a true mathematical sentence using numerals only to show the action just performed.
   [5 - 2 = 3]

5. Refer to the remainder set displayed on the table.
Ask children what they would do to show the original set.

[Put back the set of objects we took away.
Put back the pencil and the plant.
Put back Set B, etc.]

6. Ask children:

What do we call this operation on sets? [Union]

To record the action using set notation. [$C \cup B = A$]

To record "$C \cup B = A$" showing that a number property is involved. [$N(C) + N(B) = N(A)$]

To remove set B from set A again.
To record the removal of subset B showing the number property of the sets. [$N(A) - N(B) = N(C)$]

7. Children show these actions using equations

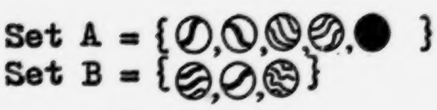
$3 + 2 = 5$
$5 - 2 = 3$

Relate Finding the Difference to One-to-One Correspondence For Comparison-Difference Subtraction

1. Present a problem:

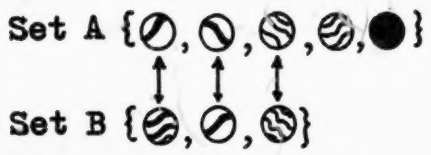
Frank has a set of marbles which we will call Set A.
Jerry has a set of marbles which we will call Set B.
We want to compare the two sets to find how many more or less one set has than the other.

Record these in set notation.



2. Tell children we will compare the elements of Set A with the elements of Set B to find the number of elements that match.

Record:



Ask children: How many elements in Set A?

Record as children respond: [N(A) = 5]

Ask children: How many elements in Set B?

Record as children respond: [N(B) = 3]

Ask children: How many elements are matched?

Record as children respond:
[3 elements of Set A are matched with 3 elements of Set B]

How many elements are not matched? Which set has more elements? Which set has fewer? What is the difference between the number of elements in Set A and the number of elements in Set B?

Direct children to show this action:

Using Set Notation [N(A) - N(B) = N(C)]

Using Numerals only [5 - 3 = 2]

3. Direct children:

Find a new set which when joined to Set B will form a set all of whose elements will match all of the elements in Set A. Call this new set, Set C.

How many elements are in Set C?

Write a mathematical sentence that shows the action of finding how - many - more are needed. [3 + ☐ = 5]

Write a mathematical sentence to show the thinking that will solve this. [5 - 3 = 2]

Subtracting Numbers

1. Test for automatic response to basic subtraction facts.

2. Drill or re-test as indicated by results.

Present each subtraction fact with its related addition fact. (Inverse Operation)

7 + 2 = 9
9 - 2 = 7

2 + 7 = 9
9 - 7 = 2

9 + 7 = 16
16 - 7 = 9

7 + 9 = 16
16 - 9 = 7 etc.

Reaching ten by regrouping the number to be subtracted.

15 - 8 thought through as:
15 - 5 - 3, then as 10 - 3 = 7

63 - 7 thought through as:
63 - 3 - 4, then as 60 - 4 = 56

452 - 5 thought through as:
452 - 2 - 3, then as 450 - 3 = 447

3. Test children's responses to extension of facts to higher decades.

A

Remainders in the Same Decade

18 - 3 = □
27 - 4 = □
39 - 7 = □

29 - 5 = □
96 - 3 = □
24 - 2 = □

B

Subtracting from Whole Decades

60 - 7 = □
40 - 8 = □
20 - 3 = □

50 - 6 = □
70 - 2 = □
90 - 4 = □

C

Remainders in the Preceding Decade

74 - 6 = □
75 - 6 = □
25 - 9 = □

51 - 9 = □
24 - 5 = □
73 - 6 = □

D

Extensions to Hundreds

374 - 3 = □
475 - 4 = □
525 - 5 = □

E

Extensions to Thousands

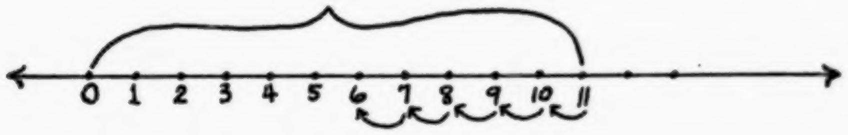
1327 - 3 = □
2440 - 4 = □
1623 - 6 = □
1603 - 5 = □ etc.

4. **Provide additional practice using varied forms.**

$235 - 8 = 230 - \square$
$2134 - 7 = 2130 - n$
$174 - n = 170 - 3$
$1405 - 8 = 1400 - \square$

5. **The number line is an effective way of providing practice. For example:**

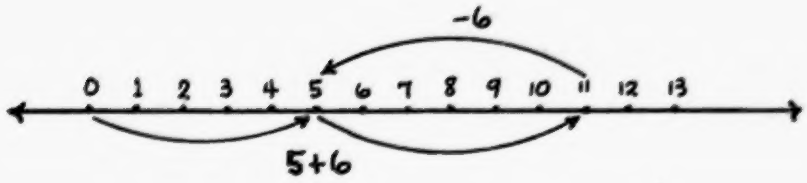
a. $11 - \square = 6$


0
1
2
3
4
5
6
7
8
9
10
11

(Optional) * b. **Use the same number line. Ask children to solve:**

$6 - 11 = \square$

c. $5 + 6 = \square$
$11 - 6 = \square$

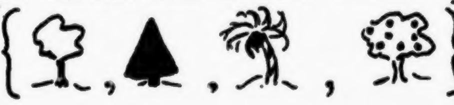

-6
0
1
2
3
4
5
6
7
8
9
10
11
12
13
5+6

6. Present true and false mathematical sentences.
Children indicate which are true and which are false, and why?
Ask them to change false sentences to true sentences.

8 - 3 = 5 because 7 + 2 = 5	[F. 8 - 3 = 5 because 5 + 3 = 8]
16 - 9 = 7 - 0	[T. because 16 - 9 = 7; 7 - 0 = 7; 7 = 7]
48 + 9 = 57 - 9	[F. because it is not an equality]
48 + 9 = 57 is the inverse operation of 57 - 9 = 48	[T.]
317 - 9 = 317 - 7 - 3	[F. because 308 does not equal 307. To make it true, the sentence must read 317 - 9 = 317 - 7 - 2]

EVALUATION AND/OR PRACTICE

1. Record in set notation a set whose elements are 4 trees.

[{  }]

Call it Set A and associate the number four with this set. [N(A) = 4]

Record any subset of this set. Call this subset Set B.
[Set B = { } (Answers may vary)]

Associate the number with Set B. [N(B) = 1 or . . .]

Find the remainder set. Call it Set C. Record the number of the remainder set.

[Set C = { }
N(C) = 3 or . . .]

2. Complete the following chart:

The number of the Original Set	The number of the Subset removed	The number of the Remainder Set
19	0	_____
49	5	_____
89	7	_____
19	_____	_____
39	_____	_____

3. Complete the following:

   In Set F there are 14 chairs.
   G is a subset of F and there are 8 armchairs in set G.
   Remove the elements of subset G.
   Show, using set notation, the number of elements in the remainder set. [N(H) = 6]

4. Additional problems may be found in textbooks.

SETS; NUMBERS; NUMERATION

UNIT 9 - SYSTEMS OF NUMERATION: BASE 10; EXPANDED NOTATION

NOTE TO TEACHER

A system of numeration must be distinguished from a number system.

A number system involves:

Ideas of quantity
Operations on the numbers
The properties that apply to those operations.

Addition and Multiplication are the major operations.

On the other hand, a system of numeration involves:

A set of symbols
Rules for using the symbols to represent and name numbers.

Common Characteristics of Systems of Numeration

1. Only a finite number of different symbols are used.
2. Any of these symbols or combinations of symbols may be repeated, and may in different positions represent different numbers.

The decimal or base ten system of numeration is only one of many systems although it is the one most commonly used.

The Hindu-Arabic or Decimal System of Numeration

Symbols

The decimal system uses ten symbols - 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

It is a place value system. The number, ten, represented by a 1 and 0 in fixed places plays a special role. Ten is called the base of this system.

The symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are called digits.

Place Value:, Grouping By Ten; Multiplicative Aspect.

Place value is the property of a system of numeration which assigns a value to a digit according to its ordered place in the numeral.

In the base-ten numeral 11, the place value of the 1 on the right is 1 one; the place value of the 1 on the left is 1 ten.

In the base-ten numeral 32, the 2 indicates a grouping of 2 ones; the 3 indicates a grouping of 3 tens.

The place of each digit represents a value ten times as great as the value of the place of the digit immediately to its right.

The place of each digit represents a value one tenth as great as the place of the digit immediately to its left.

For the numeral 1111 the value of the digits can be shown as:

10 x 10 x 10 Thousand	10 x 10 Hundred	Ten	One
100 x 10 x 1	10 x 10 x 1	10 x 1	1 x 1

For the numeral 4,736 the value of the digits can be shown as:

4 x 10 x 10 x 10	7 x 10 x 10	3 x 10	6 x 1

Expanded Notation: Additive Aspect

Expanded notation is a way of showing clearly the value of a number by adding the values represented by each digit. For example: 132 may be shown in expanded notation as:

$$132 = 100 + 30 + 2$$

$$132 = (10 \times 10) + (3 \times 10) + 2$$

$$132 = (1 \times 10 \times 10) + (3 \times 10) + (2 \times 1)$$

Role of Zero As a Digit

Zero is one of the digits, just as 1, 2, 3...9 are. As such, it can be considered a placeholder to show that there are no ones or no tens, etc. in a representation of a number.

For example, in the numeral 203, zero indicates the absence of tens.

TEACHING SUGGESTIONS

Objective: To emphasize the relationship of tens to ones, hundreds to tens, thousands to hundreds, etc.

Procedure

1. Test children's understanding of place value.

Which numeral has the digit 6 in the hundred's place?

186 603 560 3256

What number does the 8, in each of the following, represent?

871 80 1208

Write the numeral for the number that is ten more than 960.

What is the largest number which can be represented by using each of these digits $\{3, 5, 8, 4\}$ only once?

In writing the numeral for two thousand nine hundred three, in which place do we write the zero?

In which of these numerals does 5 represent half of one hundred?

750 45 500 50000

If the 3 in each of the following were changed to 7, which of the numbers would be increased the greatest?

2038 6318 3021 7324

Replace the placeholder to make a true sentence.

$300 + n + 7 = 597$

$268 = n + 8$

$3405 = 3000 + n + 5$

$n + 600 + 50 + \square = 4651$

Mark the numeral on each line that is closest in value to the one in column A.

A			
600:	200	400	900
300:	325	270	288
450:	430	500	410

2. Develop Place Value Through 10,000

Use Squared Material when necessary.
To represent 1000, prepare a strip of 10 "Hundred-Squares" taped together. Several of these strips may be used to represent larger numbers. Relate materials to the written symbols.

Start with 1000 and gradually increase to larger numbers, e.g., 1000 - 2000; 2000 - 3000; etc.

Caution: Use only teachers' demonstration materials (1") or only children's materials (1/2"). Do not combine teachers' and children's Squared Materials when representing a number.

Stress the value of the digits, e.g., 3475 as:

3000 + 400 + 70 + 5

or as

Thousands	Hundreds	Tens	Ones
3	4	7	5

Emphasize those concepts with which children have difficulty e.g.,

1001, 1010, 1101, 2001, 2010, 2101, etc.

3. Give children practice in using expanded notation.

373 = 300 + n + 3   n + 600 + 50 + 1 = 4651
136 = n + 6   2435 = 2400 + n + 5
639 = n + 30 + 9   3000 + 600 + n + 5 = 3605

SETS; NUMBERS; NUMERATION

UNIT 10 - SET OF FRACTIONAL NUMBERS: CONCEPTS AND COMPARISONS; EQUIVALENTS; COUNTING

NOTE TO TEACHER

All previous topics in this bulletin dealt with the set of whole numbers.

Number lines have been marked at selected points which correspond to whole numbers.

Now we wish to name points between those represented by whole numbers.

The number system must be extended to provide for more accurate measurement and to help deal with division which is not closed within the set of whole numbers.

The extended set of numbers is called the Set of Fractional Numbers.

A fraction can be considered either as a number or as a numeral.

Any whole number may be expressed in fraction form. For example: $\frac{2}{2}$, $\frac{0}{1}$, $\frac{9}{3}$, etc.

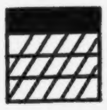
A fraction is the indicated division of an ordered pair of numbers where the second member of the pair is not zero. The second number of the ordered pair is called the denominator. The denominator indicates the number of parts of the same size into which a unit has been divided.

The first number of the ordered pair is called the numerator. The numerator indicates the number of parts of the same size being considered.

A fraction may be interpreted in several ways.

1. As one or more parts of the same size into which a unit has been divided. For example:

   $\frac{1}{4}$, $\frac{3}{4}$

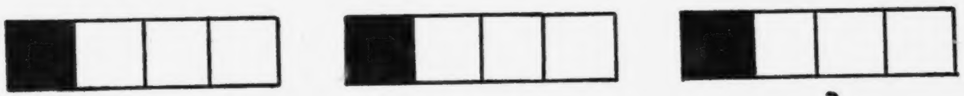


2. As a comparison or ratio. For example:

   3:4 or 3 out of 4

3. As one of each of the parts of the same size into which more than one unit has been divided. For example:

   $\frac{3}{4}$ meaning $\frac{1}{4}$ of 3 or one of each of the parts of the same size into which 3 units have been divided.



4. As an indicated division. For example: $\frac{3}{4}$ meaning $3 \div 4$.

Fractions can be represented in the following forms:

The (common) fractional form, $\frac{7}{10}$
The decimal form 0.7

The decimal form is used for convenience when the denominator is ten or a power of ten.

Mixed form is the name given to the symbol for a number involving a whole number and fraction. For example:

$3\frac{3}{4}$, 3.75

Children have used physical models and representative materials to develop fraction concepts. They have worked with units such as line segments, circular and rectangular regions. They have worked with collections of things. They have developed concepts of equivalency by comparing regions of the same size and shape.

The study of fractional parts in Grade 5 extends the application of comparisons and equivalencies.

Equivalent fractions are thought of as different symbols or names for the same fractional number.

Physical models and the number line are used to introduce new concepts and where necessary, to reinforce previous learnings.

Children who need representative materials should prepare a kit (paper discs or commercial fractional parts) of the following: 2 wholes, 4 halves, 8 fourths, 16 eighths, 6 thirds.

TEACHING SUGGESTIONS

Objective: To reinforce and extend concepts of thirds, halves, fourths, eighths

Procedures

1. Reinforce meaning of symbols
   Begin with $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{3}$

Ask children, referring, for example, to the above fractions
What does the denominator 2, 4, 8, 3 tell us?
What does the numerator tell us?
Which is larger, $\frac{1}{3}$ or $\frac{1}{2}$; $\frac{1}{3}$ or $\frac{1}{4}$? Why?

For $\frac{2}{3}$, $\frac{3}{4}$, $\frac{2}{8}$, $\frac{4}{8}$, $\frac{5}{8}$, etc. ask children:

What does the denominator 4, 8, 3, etc. tell us?
What does the numerator 2, 3, 4, etc, tell us?

Which is larger in value, $\frac{3}{8}$ or $\frac{5}{8}$? Why?

To read the following: $\frac{2}{4}$, $\frac{3}{8}$, $\frac{2}{2}$, $\frac{7}{8}$, etc.

To write, using numerals: two fourths, eight eighths, two thirds, etc.

Continue with mixed form. Ask children:

To read: $2\frac{3}{4}$, $3\frac{1}{8}$, $1\frac{7}{8}$, etc.

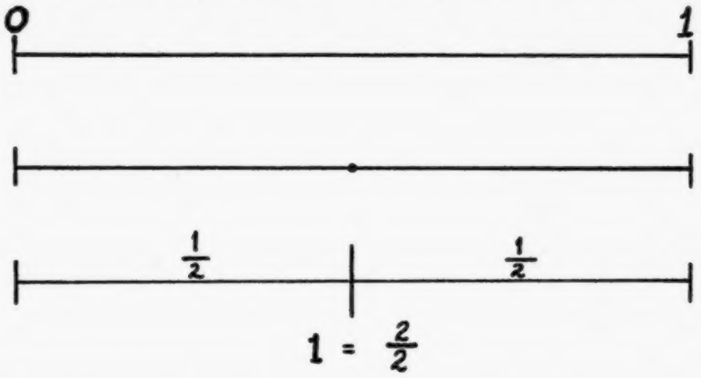
To write numerals for

one and one-half	two + three fourths
two plus three eighths	four and one fourth, etc.

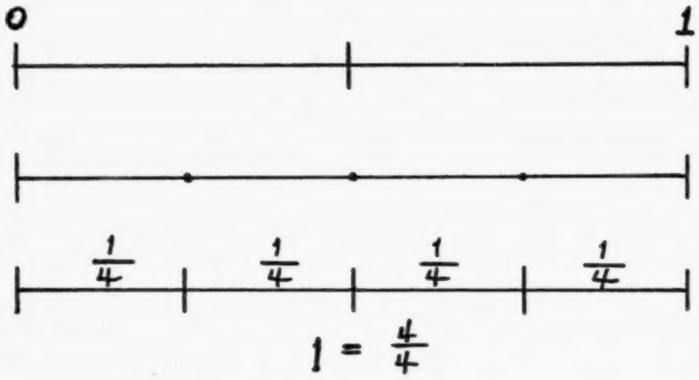
Emphasize that $2\frac{3}{4}$ means 2 plus $\frac{3}{4}$

2. Use line segment diagrams for reinforcement of concepts involving one unit.

Children draw a line segment to represent 1.
Ask children: How many halves may be shown on this line segment?



After line diagrams for halves have been developed children suggest ways to indicate fourths on a line segment.

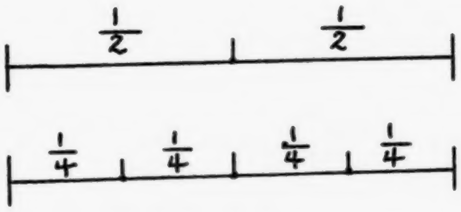


Discuss dividing the line into halves first, then each half into two equal parts. This reinforces the concept that one fourth is one half of one half.

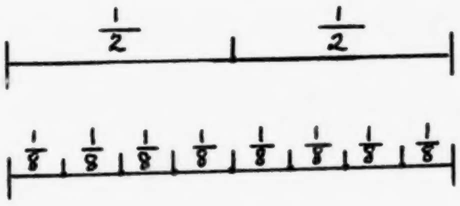
Extend to finding eighths on a line segment.
Divide line segment into halves, then into fourths, finally into eighths. Thus, one eighth is one half of one fourth, or one fourth of one half.

Children use line diagrams and rulers graduated in 8ths to deepen concepts of:

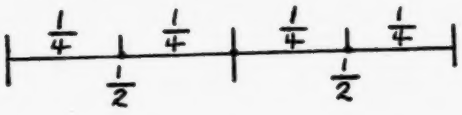
Halves and Fourths



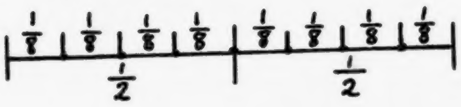
Halves and Eighths



Fourths and Halves



Eighths and Halves



Children mark the following true or false. They make false statements true.

$\frac{3}{8} = \frac{1}{4} + \frac{1}{6}$

$\frac{3}{8} = \frac{1}{2}$ of $\frac{6}{8}$

$\frac{3}{8} = \frac{4}{8}$ minus $\frac{1}{6}$

$\frac{6}{8} = 2$ times $\frac{3}{8}$

$\frac{6}{8} = \frac{1}{2} + \frac{2}{8}$

Children complete the following operations.

$\frac{1}{2} + \frac{1}{8} = \frac{\square}{8} + \frac{1}{4}$

$\frac{3}{8} + \frac{1}{8} = \frac{\square}{8} + \frac{\square}{8}$

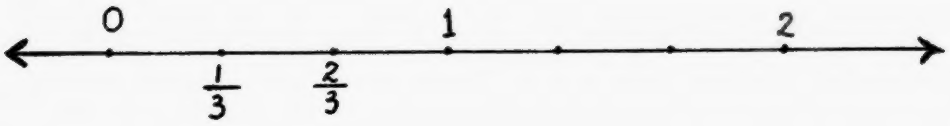
$\frac{3}{8} + \frac{1}{8} = \frac{\square}{4} + \frac{\square}{4}$

$\frac{3}{8} + \frac{1}{8} = \frac{\square}{4} + \frac{1}{8} + \frac{1}{8}$

Emphasize the fact that equivalent fractions are different names for the same number.

Wholes and thirds

Children rename 1 and 2 on the number line below as thirds.



Halves and Fourths

Children indicate that:

1 half has the same value as 2 fourths

$\frac{1}{2}$ = ? times $\frac{1}{4}$

$\frac{1}{4}$ has the same value as $\frac{1}{2}$ of $\frac{1}{2}$

$\frac{1}{4}$ is what part of $\frac{1}{2}$?

Halves and Eighths

Children replace the "n" with the correct numeral.

$\frac{2}{2} = \frac{n}{8}$, $\frac{4}{2} = \frac{n}{8}$, $\frac{8}{2} = \frac{n}{8}$

$\frac{1}{2} = \frac{\square}{8}$, $\frac{3}{2} = \frac{n}{8}$, $\frac{6}{2} = \frac{n}{8}$

$\frac{1}{2}$ = n times $\frac{1}{8}$; $\frac{1}{8}$ = n part of $\frac{1}{2}$; etc.

Fourths and Eighths

Children find solutions to the following:

$\frac{4}{4} = \frac{\square}{8}$, $\frac{5}{4} = \frac{n}{8}$, $\frac{6}{4} = \frac{n}{8}$

$\frac{2}{4} = \frac{\Box}{4}$, $\frac{4}{4} = \frac{n}{8}$, $\frac{8}{4} = \frac{n}{8}$

$\frac{1}{4}$ = ? times $\frac{1}{8}$; $\frac{1}{8}$ is what part of $\frac{1}{4}$?

$\frac{3}{4}$ = ? times $\frac{3}{8}$; $\frac{3}{8}$ is what part of $\frac{3}{4}$?

Halves, Fourths and Eighths

Children note that:

$\frac{4}{8}$ is equivalent to 1 half or 2 fourths

$\frac{6}{4}$ equals:
1 whole and 2 fourths, or
1 whole and 1 half, or
1 and 1 half, or $1\frac{1}{2}$

Children substitute numerals for n and □ to make the following equations true. (n and □ represent different numerals)

$\frac{3}{4} = \frac{1}{n} + \frac{1}{\Box}$ $\frac{5}{8} = \frac{1}{2} + \frac{\Box}{8}$

$\frac{3}{4} = \frac{n}{2} + \frac{\Box}{8}$ $\frac{5}{8} = \frac{3}{8} + \frac{\Box}{4}$

$\frac{3}{4} = \frac{2}{\Box} + \frac{1}{n}$ etc.

$\frac{3}{4} = \frac{n}{8} + \frac{1}{\Box}$

Children make these statements true.

$1 = \frac{\Box}{2} = \frac{\Box}{3} = \frac{\Box}{4} = \frac{\Box}{8}$

$\frac{1}{2} = \frac{\Box}{4} = \frac{\Box}{8}$; $\frac{\Box}{8} = 1$; $\frac{\Box}{4} = 1$; $\frac{6}{8} = \frac{\Box}{4}$; $\frac{2}{8} = \frac{\Box}{4}$

$\frac{8}{8} = \frac{\Box}{4}$; $\frac{3}{8} + \frac{1}{4} = \frac{n}{8}$; $\frac{5}{8} = \frac{1}{2} + \frac{n}{8}$; $\frac{5}{8} = \frac{1}{8} + \frac{1}{n}$

4. Comparisons - thirds, halves, fourths, eighths

Children compare: 1 whole divided into fourths, with wholes of same size divided into thirds, eighths, etc.

Children discover that:

The more equal parts into which the whole has been divided, the smaller the parts will be.

The fewer equal parts into which the whole has been divided, the larger each part will be.

For each of the following ask children:
Which is larger? Which is smaller? Why?

$\frac{3}{8}$ or $\frac{3}{4}$; $\frac{1}{3}$ or $\frac{7}{8}$; $\frac{2}{3}$ or $\frac{1}{2}$; $\frac{3}{4}$ or $\frac{3}{8}$;
$\frac{7}{8}$ or $\frac{1}{3}$; $\frac{1}{2}$ or $\frac{2}{3}$; etc.

Children use symbols (>, <) to show relationships between the pairs of fractions shown above.

Children write > or < between each group of fractional numerals below to make a true statement.

$\frac{1}{3}$ $\frac{1}{4}$; $\frac{1}{3}$ $\frac{1}{2}$; $\frac{5}{8}$ $\frac{3}{4}$; $\frac{7}{8}$ $\frac{3}{4}$;
$\frac{3}{8}$ $\frac{1}{2}$; $\frac{5}{8}$ $\frac{1}{2}$; $\frac{1}{2}$ $\frac{2}{3}$; $\frac{1}{3}$ $\frac{3}{4}$;

Children solve for n in the statements below. In some cases more than one solution is possible.

$\frac{1}{2} > \frac{n}{3}$ n = ?

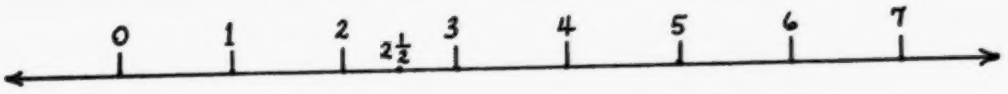
$\frac{1}{4} > \frac{n}{8}$ n = ?

$\frac{3}{4} > \frac{n}{8}$ n = ?

5. Use line diagrams and rulers graduated in eighths to reinforce or

to introduce fractional parts involving more than one unit.

Locating points on a line.



Ask children to find $1\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, $2\frac{1}{2}$, $6\frac{1}{2}$, etc.
Have them place a dot and label.

Emphasize:

One point on the line represents $2\frac{1}{2}$.

$2\frac{1}{2}$ represents the distance or length from 0 to that point.

$2\frac{1}{2}$ is 2 and $\frac{1}{2}$ more.

$2\frac{1}{2}$ means $2 + \frac{1}{2}$.

Children draw a similar line segment on which to indicate fourths.
They locate such points as:

$2\frac{1}{4}$, $4\frac{1}{4}$, $6\frac{1}{4}$, etc.

$\frac{2}{4}$, $3\frac{2}{4}$, $1\frac{2}{4}$, $5\frac{2}{4}$, etc.

$1\frac{3}{4}$, $2\frac{3}{4}$, $4\frac{3}{4}$, etc.

Discuss distance from 0 and distance between points, e.g.,

between $1\frac{3}{4}$ and $3\frac{3}{4}$

between $2\frac{1}{4}$ and $3\frac{3}{4}$

Proceed as above for eighths; thirds; halves; fourths and eighths.

Discuss:

a. Where 1 half, 1 fourth, 1 eighth are located on a number line.

b. Find on the number line:

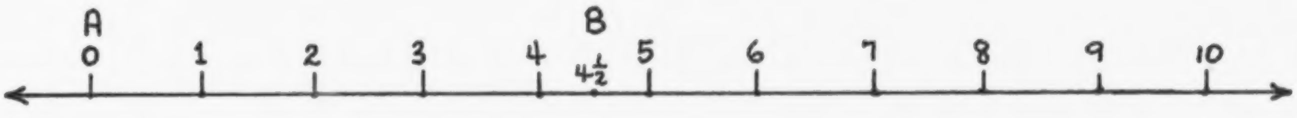
$1\frac{1}{2}$, $4\frac{1}{4}$, $5\frac{1}{8}$, $6\frac{2}{8}$, $7\frac{3}{4}$, $3\frac{5}{8}$

c. Find $1\frac{1}{2}$ on the number line. How many halves in $1\frac{1}{2}$?

Find $1\frac{1}{2}$ on the number line. How many fourths in $1\frac{1}{2}$?

6. Enrichment Activities

Present a number line.



Discuss:

Distance from A to B = distance from B to C. Find Point C.

A	B	C	
0	$4\frac{1}{2}$	?	
4	$13\frac{1}{2}$	?	
$2\frac{1}{4}$	4	?	etc.

Compare the number line with a ruler graduated in eighths.

Children find equivalents for more than 1 whole using line diagrams if needed.

A	B
$\frac{3}{2} = 1 + ?$	$2\frac{1}{4} = \frac{?}{4}$
$\frac{5}{2} = ? + \frac{1}{2}$ etc.	$4\frac{3}{4} = \frac{n}{4}$ etc.

C	D	E	
$3\frac{3}{8} = \frac{n}{8}$	$1\frac{5}{8} = \frac{n}{8}$	$2\frac{1}{4} = \frac{n}{2} + \frac{\square}{4}$	
$1\frac{5}{8} = \frac{n}{8}$	$1\frac{3}{8} = n + \frac{\square}{8}$	$3\frac{1}{2} = \frac{n}{4}$	etc.

Children relate to ruler:

Halves in 1 inch, 2 inches, 4 inches, etc.
Halves in $1\frac{1}{2}$ inches, $2\frac{1}{2}$ inches, etc.

Fourths in 1 inch, 2 inches, 3 inches, etc.
Fourths in $1\frac{1}{2}$ inches, $1\frac{1}{4}$ inches, etc.

Eighths in 1 inch, 2 inches, 3 inches, etc.
Eighths in $\frac{1}{2}$ inch, $\frac{1}{4}$ inch, $\frac{3}{4}$ inch, $1\frac{1}{4}$ inch, etc.

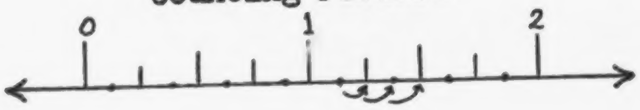
Eighths in $5\frac{1}{4}$ inches

Eighths in $3\frac{3}{4}$ inches

7. Counting

Children use fractional parts and / or number lines to count forward and backward by halves, fourths, eighths.

Counting Forward



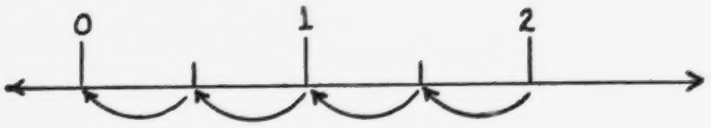
$1\frac{1}{8}$, $1\frac{2}{8}$, $1\frac{3}{8}$, $1\frac{4}{8}$, —, —, —, — Later: $1\frac{1}{8}$, $1\frac{1}{4}$, $1\frac{3}{8}$, $1\frac{1}{2}$, —, —, —, —

$2\frac{1}{4}$, $2\frac{2}{4}$, $2\frac{3}{4}$, $2\frac{4}{4}$, —, —, —, — Later: $2\frac{1}{4}$, $2\frac{1}{2}$, $2\frac{3}{4}$, 3, $3\frac{1}{4}$, $3\frac{1}{2}$, $3\frac{3}{4}$.

Children record series of fractions which they counted.

Note: Some children need to count forward and backward with simpler intervals, e.g., counting forward by fourths beginning with $\frac{1}{4}$.

Counting Backward



$\frac{4}{2}$, $\frac{3}{2}$, $\frac{2}{2}$, $\frac{1}{2}$, —, —, —, — Later: 2, $1\frac{1}{2}$, 1, $\frac{1}{2}$, etc.

$\frac{8}{4}$, $\frac{7}{4}$, $\frac{6}{4}$, $\frac{5}{4}$, —, —, —, — Later: 2, $1\frac{3}{4}$, $1\frac{1}{2}$, $1\frac{1}{4}$, etc.

$\frac{8}{8}$, $\frac{7}{8}$, $\frac{6}{8}$, $\frac{5}{8}$, —, —, —, — Later: 1, $\frac{7}{8}$, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{1}{2}$, etc.

Children record series of fractions which they counted.

Children count forward or backward by $\frac{1}{2}$ beginning with fractions other than $\frac{1}{2}$. They use number line.

$\frac{1}{4}$, $\frac{3}{4}$, $1\frac{1}{4}$, —, —, —, —, —
$3\frac{3}{4}$, $4\frac{1}{4}$, $4\frac{3}{4}$, —, —, —, —
$4\frac{1}{4}$, $3\frac{3}{4}$, $3\frac{1}{4}$, —, —, —, —, —, etc.

Children count forward by 2 eighths beginning with any point on the line.

$\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, $\frac{9}{8}$, etc. Later: $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, $1\frac{1}{8}$, —, —

They count forward by 3 fourths beginning with any point on the line.

$\frac{1}{8}$, $\frac{7}{8}$, $\frac{13}{8}$, etc. Later: $\frac{1}{8}$, $\frac{7}{8}$, $1\frac{5}{8}$, etc.

They fill in the missing numerals:

$3\frac{1}{8}$, $3\frac{3}{8}$, —, —, $4\frac{1}{8}$, —, —, $4\frac{7}{8}$

GEOMETRY AND MEASUREMENT

UNIT 11 - MEASUREMENT: LENGTH

TEACHING SUGGESTIONS

Objective: To reinforce and extend concepts of length;
Relationships among standard units of length.

Procedures

1. Continue to develop finding $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$ of a foot. Express in inches.

   Materials used: Various types of rulers; yardstick; tape measure.

   Discuss relationships:

   1 foot equals 12 inches (is the same length as)

   $\frac{1}{2}$ foot equals 6 inches

   $\frac{1}{4}$ foot equals 3 inches

   $\frac{1}{3}$ foot equals 4 inches

   $\frac{1}{3}$ yard equals 12 inches

   $\frac{1}{3}$ yard equals 1 foot

2. Extend development to include finding $\frac{1}{8}$ of a yard. Relate to $\frac{1}{4}$ of a yard. Express in inches.

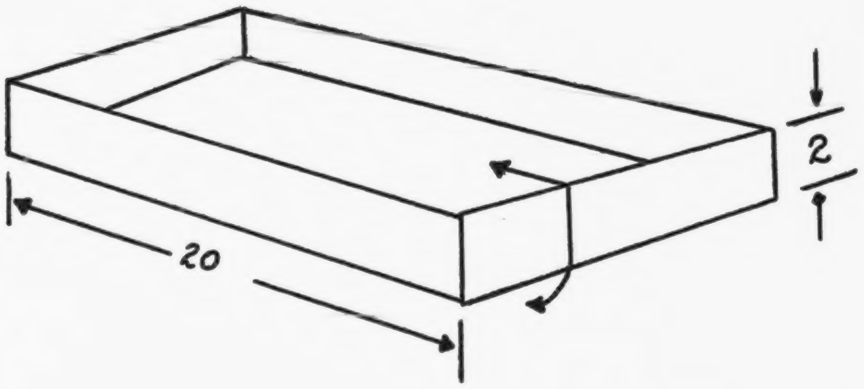
   $\frac{1}{4}$ yard = 9 inches

   $\frac{1}{8}$ yard = $\frac{1}{2}$ of $\frac{1}{4}$ yard

   $\frac{1}{8}$ yard = $\frac{1}{2}$ of 9 inches = $4\frac{1}{2}$ inches

3. Measures relating to mailing parcel post packages.

Have packages of different shapes for pupils to measure. They decide which packages can be sent by parcel post. Apply regulations: "Limit 70 pounds, measuring not more than 100 in. in length and girth combined." Girth is the distance around the package at its widest part.



e.g., Length = 20"
Girth = 44"
Combined length and girth = 64"

4. Reinforce Equivalents

Teacher and children organize a table of the fractional parts of the foot and the yard in inches and stress relationships in various ways.

Relationships among:

Yards, Feet, Inches
1 ft. = 12 in.
1 yd. = 36 in.
1 yd. = 3 ft.
1/3 yd. = 1 ft.
1/2 yd. = 18 in.

Feet and Inches
1 ft. = 12 in.
1/2 ft. = 6 in.
1/4 ft. = 3 in.
1/3 ft. = 4 in.
2/3 ft. = 8 in.

Yards and inches
1 yd. = 36 in.
1/2 yd. = 18 in.
1/4 yd. = 9 in.
3/4 yd. = 27 in.
1/8 yd. = 4 1/2 in.
1/3 yd. = 12 in.
2/3 yd. = 24 in.

There are numerous ways of reinforcing these concepts and relationships by organizing different tables in tabular form.

Encourage children to prepare their own tables showing equivalents.

Discuss alternate use of fractional parts of measurements.

We purchase $1\frac{1}{2}$ yd. of ribbon rather than 54 inches.
A carpenter may refer to a length of board as $3\frac{1}{4}$ feet
instead of 39 inches.

5. Provide practice in changing to different units.

18 in. $= 1$ ft. 6 in. or $1\frac{1}{2}$ ft. or $\frac{1}{2}$ yd.

$1\frac{1}{2}$ yd. $= 3$ ft. 18 in. or 54 in. or $4\frac{1}{2}$ ft., etc.

$\frac{3}{4}$ yd. $= \square$ in.

Since $\frac{1}{4}$ yd. $= 9$ in., then $\frac{3}{4}$ yd. $= 9$ in. $+ 9$ in. $+ 9$ in. $= 27$ in.

6. Number of feet in one mile.

Suggested activities to develop the concept of the length of a mile.

Estimate and then pace off the length of the classroom.
Estimate and pace off distances in the playground.
Pace off a block.
Locate a familiar landmark about one mile from school.

Pupils visualize the distance of a mile by:

Observing the odometer while riding on a bicycle or in a car.
Relating distance and time.
Using pedometers on walks or hikes.

Note: Children should be aware of the fact that the odometer in a car or on a bicycle registers distance in tenths of a mile.

Discuss distances

Approximate number of blocks in a mile.
Number of blocks from home to school and from school to near-by places of interest.
Bus trips from school to places of interest.

Vacation trips by car .
Heights of mountains.
Heights at which planes travel.
Lengths of rivers and lakes.
Distances run in races.
Lengths of track in athletic stadiums.

7. Have children construct charts to show distance from school to:

Post Office	11 blocks	about $\frac{1}{2}$ mile
Museum	23 blocks	more than a mile
Park	8 blocks	less than $\frac{1}{2}$ mile
Library	5 blocks	about $\frac{1}{4}$ mile

✻ 8. Discuss units of measure of the Past: (Optional)

cubit
fathom
digit
furlong
league
span
hand
pace

EVALUATION and / or PRACTICE
SUGGESTED EXERCISES

1. Tell which is more and why

8 inches or $\frac{1}{2}$ yard

$1\frac{1}{4}$ feet or $11\frac{1}{2}$ inches

2. Arrange the following units of length from the shortest to the longest unit.

yard inch foot mile

3. Draw a line segment $\frac{3}{4}$ of an inch in length.

   How many $\frac{1}{8}$ inches in this segment?

   How many $\frac{1}{4}$ inches?

4. Children answer the following:

   Ann had one yard of felt. She cut off 12 inches to make a pencil case. What part of the yard did she use?

   18 inches $\qquad \frac{1}{2}$ yard $\qquad \frac{1}{4}$ yard $\qquad \frac{1}{3}$ yard

   A piece of cloth is 38 inches long. Jane cut off 1 yard to make a towel. How much of the cloth was not used?

   The teacher cut a yard of leather into 8 equal pieces to make bookmarks. Each piece was

   $\frac{1}{2}$ yard $\qquad$ 8 inches $\qquad$ 9 inches $\qquad 4\frac{1}{2}$ inches

5. How many feet are there in a mile; how many yards; how many inches?

*Enrichment Activities **(Optional)**

1. Extend concepts of the mile

   Children report on:

   History of the development of the distance of a mile.

   [Originally the distance of a mile was established as 1000 double paces. One pace is about 3 feet.]

   Relationship of length of football field to the mile.

   [A football field is 100 yards or 300 feet in length. Twenty such fields would be about one mile.]

   Terms, such as: standard mile, statute mile,
   nautical mile, knots

2. Measurements mentioned in literature, such as:

   "Charge of the Light Brigade" - Half a league onward.

   "Bible" - Noah's Ark; Solomon's Temple

3. Have children explore possible ways to obtain the approximate height of a very tall object, such as the school flagpole.

   Hint: Compare with height of school building where each story is approximately 10 feet high

   and / or

   Child stands next to flagpole. Another child measures the length of his shadow and the length of the shadow of the flagpole. What is the relationship of the child's height and the length of his shadow.

GEOMETRY AND MEASUREMENT

UNIT 12 - MEASUREMENT: TIME

NOTE TO TEACHER

In grade 5, problems involving denominate numbers are solved by computing mentally. Problem solving should include the use of equivalents. Problems may be made by teachers using data from charts. Additional problems may be obtained from various textbooks.

TEACHING SUGGESTIONS

Objectives: To introduce finding fractional parts of time.
To help children interpret and record time.
To help children solve problems.

Procedure

The Calendar

1. Reinforce
    - Number of months in one year
    - Names of the months of the year
    - Names of the seasons
    - Number and names of the months in each season
2. Introduce
    - Finding $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$ of a year in months
    - Relating $\frac{1}{4}$ of a year to the seasons
3. Typical Problems

    Today is Sept. 25. My dental appointment is two weeks from today. What date will that be?

Halloween is on Oct. 31. Invitations for a party should be mailed ten days in advance. On what day and date shall I mail them?

(Children solve problems in a variety of ways.)

Suggested Experiences

Noting seasonal changes - weather, outdoor phenomena
Computing variations in length of daylight, length of night.
Marking data on calendar - classroom parties, assembly programs, special events, etc. - Computing length of time until event.
Days, weeks, school days in a month, etc.

Visits to dentist - twice a year, every six months, - semi-annually.
Estimate calendar time since last visit.

Ways in which to record dates - 10/3/66, 10 - 3 - 66

The Clock

1. Continue to develop:

   Minutes in one hour, half hour, quarter hour, three-quarter hour
   Telling time by the hour, half hour, quarter hour, 5 minute intervals, etc.

   A.M. and P.M. - before noon and afternoon. Extend understanding to ante and post meridian.

   A day as extending from midnight to midnight

2. Interpret and record time

Assembly Program	9:00 A.M.	
Recess	10:30 A.M.	(read as half past ten or ten thirty)
Supper	6:15 P.M.	(15 minutes after 6 or a quarter past 6 or 6 fifteen)
Sunrise	5:18 A.M.	(18 minutes after 5, or 5 eighteen)
Sunset	6:50 P.M.	(6 fifty, 10 minutes to 7)
Lunch	12:00	(12 noon)

3. Solve "verbal" problems in a variety of ways.

   How long was it from supper to sunset?
   How much more daylight did we have today than we did on Monday?

Our Trip To The Museum

Left school	9:30
Arrived at Museum	10:15
Lecture - Far North	10:25
Eskimo Exhibit	11:00
Film - Eskimo Life	12:05
Lunch	12:45

How long did it take us to get to the museum?

To solve, children think:
From 9:30 to 10:00 A.M. ($\frac{1}{2}$ hour), from 10:00 to 10:15 ($\frac{1}{4}$ hour).
The trip took $\frac{3}{4}$ of an hour.

Some children might compute in minutes:
From 9:30 to 10:00 (30 min.), from 10:00 to 10:15 is another 15 minutes. The trip to the museum took 45 minutes.

We came to school at 8:40. How much time did we have before we left for the museum?

How long was it from the time we arrived at the museum until lunch time?

We were due back at school at 3:00 o'clock. If the trip back takes 50 minutes, at what time must we leave the museum?

Some children may need to refer to a clock or a clock face.

Experiences, Activities and / or Homework

Children:

Compare variations in time of daylight, night.
Plan for time of preparation and duration of party.
Estimate amount of time for preparation of assembly programs, culmination of unit, etc. Use both calendar time and clock time.

Compute time for out-door activities
Plan schedule for the school day
Note and record time of sunset and sunrise (refer to newspaper)
Note changes in length of days
Construct and interpret charts based on information found in newspapers, e.g., time of sunrise and sunset
Note time of radio and television programs
Note time spent in various activities outside of school
Discuss best use of time at home
Use schedules and time tables (discussing travel time)

* Enrichment Activities (**Optional**)

Using Timetables and Bus Schedules

Select two trains, two buses, or a train and a bus. Determine which has the greater speed by comparing travel time between the same places.

Write a problem involving which train or bus to take, in order to keep an appointment in another city, at a specific time.

Children look up and explain

Other calendars, e.g. Gregorian

Ancient devices for telling time: Sundial; water-clock; etc.

SETS; NUMBER; NUMERATION

UNIT 13 - SYSTEMS OF NUMERATION: ROMAN SYSTEM

NOTE TO TEACHER

The Roman System of Numeration is taught for two reasons.

1. Understanding of various systems of numeration helps to deepen children's understanding of the decimal system.
2. Roman numerals are still used on clock faces, for chapter headings in books, on cornerstones of buildings, etc.

We teach the Roman system as another way of recording numbers, another system of numeration.

We should stress the relationships within the system and comparisons with the decimal (Hindu-Arabic) system of numeration.

The Roman System uses seven symbols:

I, V, X, L, C, D, M.

The following principles are used in combining these symbols to represent numbers:

1. The Rule of Repetition: When the Roman Numeral, I, X, C, M is repeated its value is added. For example: XXX represents 10 + 10 + 10.
2. The Rule of Addition: The values of the Roman Symbols are added when the symbol representing the larger number is placed at the left in the numeral. For example:

   VII = V + I + I; DC = D + C

3. The Rule of Subtraction: When a numeral of lesser value is written to the left of a numeral of greater value the lesser value is to be subtracted from the greater. See Item 2 under "Teaching Suggestions". (This is a late adaptation of the original Roman System).

   The subtractive property is used specifically to represent numbers such as: four, nine, ninety, four hundred, nine hundred.

   IV = four
   XL = forty
   CD = four hundred

   IX = nine
   XC = ninety
   CM = nine hundred

Some comparisons that can be made with the Decimal system are:

1. There is no symbol for "zero" in the Roman System of Numeration.
2. Place Value is not used in the Roman System except when the subtraction rule is involved.
3. The Rule of Repetition is not used in the Decimal System.
4. The Rule of Subtraction is not used in the Decimal System.

TEACHING SUGGESTIONS

Objectives: To introduce new symbols and their values.
To compare the Roman System of Numeration with the Decimal System of Numeration.
To provide practice in reading and writing Roman Numerals.

Procedure

1. Evaluate children's understanding of the Roman System.

   Write the following in the decimal system of numeration:

   VII XXIX XXXV CCII
   XIV XLVI XLIX CXIV

Write the following using Roman numerals:

8	41	24	94
17	39	50	140

Write a Roman numeral of two symbols which shows that the value of one symbol is to be added to the value of the other symbol to obtain the value of the entire written symbol.

Write a Roman numeral of two symbols which shows that the value of one symbol is to be subtracted from the value of the other symbol to obtain the value of the entire written symbol.

Choose the Roman numeral on each line which is the sum for the additions at the left:

IV + VI	VVII	IVVI	X
IX + XI	XX	IXXI	XXII
XXI +XIX	XXXI	XXIXIX	XL
XXV +XXV	XXXX	L	XXXVV

2. Reinforce understanding of the Roman System

Discuss the value of the basic symbols (I, V, X, L, C) used in the Roman system.

Have children compare the Hindu-Arabic system of notation, which is a positional system, with the Roman system.

Be sure children understand that:
When a Roman numeral is repeated, its value is added.

II represents 1 + 1 or 2
III represents 1 + 1 + 1 or 3
XX represents 10 + 10 or 20
XXX represents 10 + 10 + 10 or 30
CC represents 100 + 100 or 200

Roman symbols "V" and "L" are not repeated, since the value of two V's is represented by X and the value of two L's is represented by C.

When a Roman numeral of lesser value is written before a numeral of greater value, it indicates subtraction of the lesser value from the greater value. The symbols V, L are never written to the left of numerals of greater value

I to the left of V (IV) indicates 1 less than 5
I to the left of X (IX) indicates 1 less than 10
X to the left of L (XL) indicates 10 less than 50
X to the left of C (XC) indicates 10 less than 100

The symbol I may be used subtractively with V and X only.

The symbol X may be used subtractively with L and C only.

3. Continued Development

Test children's ability to translate Roman numerals into Arabic numerals and Arabic numerals, through 399.

Introduce D as the Roman numeral representing 500. Have children write the Roman numerals for 390 **through** 500.

Provide practice in reading and writing Roman numerals which represent numbers through 500 and beyond.

EVALUATION and / or PRACTICE

SUGGESTED EXERCISES

1. Children read the following:

V	VI	IV	VIII	Etc.		
X	IX	XI	XXI	XV	XXX	Etc.
L	XL	LX	LXIX	XLV	LVII	Etc.
C	CCX	XC	CIX	CXI	CL	Etc.

2. **Children write the following series using Roman numerals.**

10 - 19	89 - 95	310 - 319
40 - 49	95 - 106	500 - 508
70 - 80	281 - 292	590 - 600

✱ 3. Point out the advantage of the Decimal System. **(Optional)**

✱ 4. Explore one or two computations involving addition. and / or multiplication using Roman numerals. **(Optional)**

✱ 5. Explain why the Romans did not include a symbol for zero in their numerals. **(Optional)**

OPERATIONS

UNIT 14 - SET OF WHOLE NUMBERS: ADDITION AND SUBTRACTION; PROPERTIES APPLIED; HORIZONTAL FORMAT

NOTE TO TEACHER

Adding and subtracting by groups (mental computation) involves computing from left to right rather than from a visualized vertical algorithm.

This generally involves the application of the Associative Property for Addition. Children should recognize the need for quick mental computation in real life situations.

Encourage children to compute "mentally" with the numbers that they can deal with easily.

Encourage them to discover many ways to arrive at sums and remainders. There is no one correct or best way of arriving at a solution. For example:

To add:

58 + 26: A child might begin with the first addend and rename 26 as 20 + 2 + 4. He could then apply the associative property and think 58 + 20 = 78, 78 + 2 = 80, 80 + 4 = 84.

or

A child might rename 58 as 50 + 8 and rename 26 as 20 + 6. He could then apply the associative property and think 50 + 20 = 70, 8 + 6 = 14, 70 + 14 = 84

or

A child might think 60 + 26 = 86, 86 - 2 = 84, etc.

To subtract 192-29:

A child might think: 192-20 = 172, 172-9 = 163 or

A child might think: $192-30 = 162$, but since I subtracted 1 too many, I must add 1; $162 + 1 = 163$.

To compute using a horizontal format children need to have automatic response to basic addition and subtraction facts, and have ability in adding and subtracting numbers involving extensions beyond the facts. They need facility in regrouping numbers.

In presenting addition or subtraction exercises:

Record the entire exercise

or

Write one of the numerals and present the other orally

or

State the entire exercise orally.

Children record only the sum or the remainders.

Teacher may wish to write some of the partial sums or remainders as the child states the thinking.

TEACHING SUGGESTIONS

Objective: To develop facility in adding and subtracting "mentally".

Procedures

1. Test children's understanding of relationships.

Children replace "n" with numerals, to make the equation a true statement.

Use of Associative Property	Indirect Application of Associative Property
$45 + 23 = 65 + n$	$68 - 23 = 48 - n$
$62 + 19 = 72 + 8 + n$	$92 - 57 = 42 - n$
$58 + 36 = n + 6$	$81 - 45 = n - 5$
$324 + 150 = 424 + n$	$237 - 12 = 227 - n$

Doubles and Near-Doubles

$49 + 50 = 100 - n$
$250 + n = 500$
$125 + 126 = 250 + n$
$86 + 86 = 160 + n$
$57 + 57 = n + 14$

Use of the Commutative Property

$32 + 265 = 265 + n$
$n + 458 = 458 + 24$
$21 + 518 = n + 21$
$57 + n = 112 + 57$

2. Provide practice in adding and subtracting using the horizontal format. From time to time present addition and subtraction exercises as shown below to emphasize that each addition fact has a related subtraction fact.

Have children explain this relationship. (Idea of inverse)

$66 + 7 = \square$
$236 + 7 = \square$
$196 + 7 = \square$
$2256 + 7 = \square$
$1396 + 7 = \square$

$73 - 7 = \square$
$243 - 7 = \square$
$203 - 7 = \square$
$2263 - 7 = \square$
$1403 - 7 = \square$

The following suggested exercises are graded as to difficulty. Children should find sums and remainders. The teacher should ask some children to explain how they arrived at a solution. They show how the Commutative and Associative Properties are being used to justify steps in the computation.

A

$27 + 27 = \square$
$65 + 65 = \square$
$45 + 18 + 26 = \square$
$12 + 36 + 27 + 23 = \square$
$67 + 61 = \square$
$56 + 57 = \square$
$34 + 85 + 26 = \square$

$80 - 39 = \square$
$99 - 14 = \square$

B

$243 + 34 = \square$
$243 + 38 = \square$
$507 + 68 = \square$
$40 + 319 = \square$

$270 - 30 = \square$
$567 - 60 = \square$
$184 - 32 = \square$
$364 - 45 = \square$

C

$140 + 140 = \square$
$122 + 122 = \square$
$125 + 125 = \square$
$115 + 115 = \square$
$135 + 135 = \square$

$220 - 160 = \square$
$290 - 145 = \square$
$350 - 175 = \square$

3. Extend development to numbers through 9999. Present the following graded series discussing the problems and ways of solving them. Most children will need to work with smaller numbers first.

Present many more examples of each type within a series. (Read 4-place numerals as hundreds to facilitate arriving at sums and remainders).

A

2300 + 3000 = □
1160 + 1000 = □
3226 + 2000 = □
5448 + 4000 = □

5300 - 3000 = □
2160 - 1000 = □
9448 - 4000 = □
7008 - 2000 = □

B

2416 + 60 = □
2006 + 60 = □
3652 + 20 = □
5048 + 30 = □

2476 - 60 = □
4870 - 40 = □
1396 - 50 = □
5078 - 30 = □

C

1538 + 200 = □
3215 + 300 = □
4351 + 100 = □
3642 + 200 = □

5117 + 700 = □

D

2760 - 200 = □
1938 - 400 = □
4651 - 300 = □
3842 - 500 = □

5617 - 700 = □

E

1163 + 15 = □ [May be thought through as (1163, 1173, 1178)]
2434 + 23 = □
62 + 4526 = □ [May be thought through as 4526 + 62]
3058 + 41 = □
5555 + 32 = □

1178 - 15 = □ [May be thought through as (1178, 1168, 1163)]
2457 - 23 = □
3149 - 37 = □
4189 - 22 = □
5276 - 43 = □

F

2536 + 24 = n [Thought through as (2536, 2556, 2560)]
1769 + 25 = n
1441 + 39 = n
2657 + 18 = n
3223 + 49 = n

3244 - 26 = n [Thought through as (3244, 3224, 3220, 3218)]
2560 - 34 = n
1480 - 39 = n
2675 - 18 = n
3272 - 47 = n

OPERATIONS

UNIT 15 - MULTIPLICATION OF WHOLE NUMBERS: FACTS AND EXTENSIONS BEYOND THE FACTS

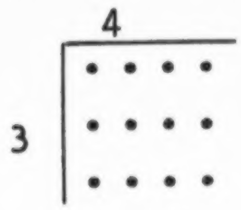
NOTE TO TEACHER

Interpretation of Multiplication

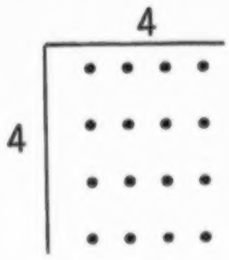
In Addition, an ordered pair of numbers is operated on to yield a third number called their sum.

In Multiplication, a unique third number, called the product, is similarly assigned to an ordered pair of numbers.

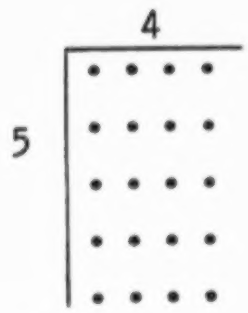
There are at least two interpretations involving the operation of multiplication on the set of whole numbers. In previous grades children have interpreted multiplication as repeated addition. This interpretation is shown below by the use of rectangular arrays. An array is an orderly arrangement of sets of things, numerals or other symbols.
Consider these arrays:



3 x 4 = 12
or
4 x 3 = 12



4 x 4 = 16



5 x 4 = 20
or
4 x 5 = 20

In each of the arrays above the product may be computed by successive additions.
Thus, the product for 3 x 4 may be interpreted by considering 4 as an addend 3 times; 4 + 4 + 4 = 12 or 3 as an addend 4 times as 3 + 3 + 3 + 3 = 12.

Concept of Product Set

Another interpretation of multiplication of whole numbers involves a situation such as the following:

> Joan has a set of 3 new blouses;
> a red one, a yellow and a white one.
> She has a set of 2 new skirts;
> a grey one and a blue one.
> She wants to know how many different
> outfits she can have.

Possible matchings:

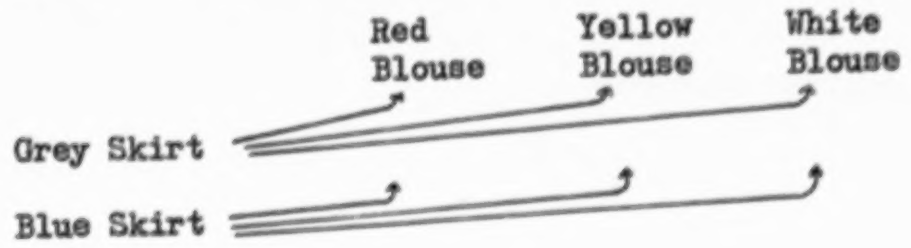
Grey skirt, red blouse
Grey skirt, yellow blouse
Grey skirt, white blouse

Blue skirt, red blouse
Blue skirt, yellow blouse
Blue skirt, white blouse

The Product Set (also called Cartesian Product) of two sets A and B is denoted by A x B and is defined as the set of all ordered pairs (a,b) where a is an element of Set A and b is an element of Set B.

Here if A = {red, yellow, white} and
B = {grey, blue}
then the six ordered pairs above list the elements of A x B.

Notice that $N(A \times B) = 6$
$N(A) = 3$
$N(B) = 2$
and that $N(A \times B) = N(A) \times N(B)$ which is the reason why A x B is called the product set.
Thus A x B in the above illustration can be represented on a chart as:



and represented as an array:

			Blouses	
		Red	Yellow	White
Skirts	Grey	•	•	•
	Blue	•	•	•

There are then, six matchings possible and we see that by this interpretation of multiplication of the ordered pair (2, 3) we arrive at the same array with 6 elements as in the first interpretation.
The numbers operated on, 3 and 2, are called factors. The result of the operation, 6, is called the product.

Factor x Factor = Product

Properties of Multiplication

Some of the properties that apply to multiplication and which will be used in this bulletin are.

Commutative Property for Multiplication which states that reversing the order of the factors does not affect the product.
For example: $2 \times 3 = 3 \times 2$

Associative Property of Multiplication which states that the order of associating a product involving more than two factors does not affect the product.
For example:

$$6 \times 5 = (3 \times 2) \times 5 = 3 \times (2 \times 5) = 3 \times 10$$

Distributive Property of Multiplication With Respect to Addition combines multiplication and addition.
For example:

For 6×23, we can write

$$6 \times (20 + 3) = (6 \times 20) + (6 \times 3)$$

For 23×6, we can write

$$(20 + 3) \times 6 = (20 \times 6) + (3 \times 6)$$

and we obtain the same product in both cases, because of commutativity.

Identity Element for Multiplication

The number "One" has a special property with respect to multiplication. The product of one and any number is that same number.
For example: 1 x 8 = 8; 8 x 1 = 8

Multiplicative Property of Zero

The product of zero and any number is zero.
For example: 0 x 8 = 0; 8 x 0 = 0; 0 x 0 = 0

Closure

The set of whole numbers is closed with respect to multiplication; that is, when multiplying any two whole numbers, the product is always a whole number.
For example: 8 x 4 = 32

Drill on Multiplication Facts and Extensions

Beyond the Facts

Multiplication and division facts were developed in Grade 4. However, drill for automatic response is still essential in Grade 5.

After an interval of meaningful drill, and/or, development, the teacher should test for automatic response.

Keep drill periods short (5 - 10) minutes.
Children's responses may be written or oral.
Written responses are preferable.
Children record products or quotients only.

Present sets of multiplication facts according to specific patterns or relationships. One or more of these patterns may be presented during the drill period. Guide cards showing these patterns may be used by the teacher for easy reference.

Equations stressing properties of addition and multiplication can be used as another form of drill. These, too, should be organized according to specific patterns.

Relate the drill to the major topic to be developed wherever possible.

Extension beyond the basic facts are needed for mental computation in multiplication and as background for division.
Sentences should be presented in horizontal form.
The teacher may record facts using words and numerals or numerals only, e.g. 2 sixes or 2 x 6.
Children may read 3 x 6 as 3 sixes or 6 threes (3 - six times).
Note that, because of commutativity of multiplication we no longer give just one interpretation to 3 x 6.

TEACHING SUGGESTIONS

Objectives: To help children extend their concepts of multiplication through the use of arrays.

To teach the concept of factors, multiples.

To reinforce multiplication facts and extensions.

Arrays

1. Present a problem such as:

   Joan has a set of 3 new blouses; a red one, a yellow and a white one. She has a set of 2 new skirts; a grey one and a blue one. She wants to know how many different outfits she can wear.

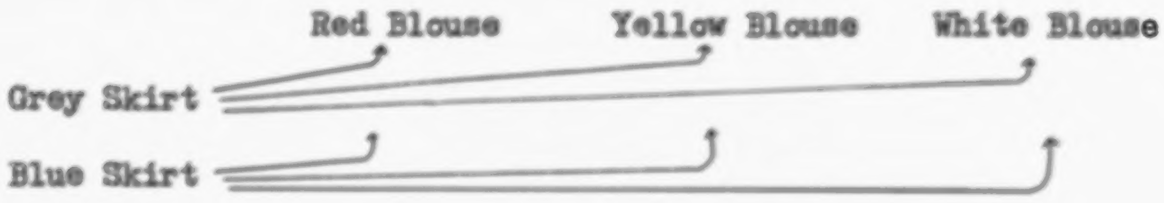
   Ask children:

   In how many different ways can Joan match a blouse and skirt?

   Record the different matchings as children discuss them.

   Grey skirt, red blouse
   Grey skirt, yellow blouse, etc.

2. Teacher and children should prepare a chart on the chalkboard showing the different matchings.



3. Tell children that what we have shown is called an array. The array is also shown in another way in the Note to Teacher, page

   Ask children:

   How many different blouses can Joan wear with her grey skirt? [3; red, yellow, white]

   How many different blouses can she wear with her blue skirt? [3; red, yellow, white]

   How many different skirts can Joan wear with her red blouse? [2; grey, blue]
   with her yellow blouse? [2]
   with her white blouse? [2]

   How many different outfits can Joan make matching 2 skirts and 3 blouses? [6]

   How can we determine the number of different outfits Joan can have without preparing an array? [2 x 3 or 3 x 2]

   How many outfits would Joan have if she had 4 blouses and 2 skirts?

   Tell children to examine the array and ask them whether they can tell what an array is. Use a multiplication chart to show an array of numerals. Elicit from them that an array is an orderly arrangement of objects or numerals.
   Point out that here we use rectangular arrays.

4. Present another problem:

   Three boys (John, Ted and Bill) and 3 girls (Mary, Sue and Alice) came to the school party. Draw an array to show how many different sets of partners there can be for dancing.

Direct children:

To make an array to show the matchings in the problem.

To tell how they can find the solution without counting the dots in the array.
[Multiply the number of rows by the number of columns]

To tell how they would find the solution without drawing an array. [Multiplication]

5. Ask children to draw an array to show 4 x 5.
Discuss the multiplication facts chart as an array.

x	0	1	2	3	etc
0	0				
1					
2					
3					
etc					

*6. Ask children to draw an array to show 1 x 5; to show 0 x 5 (Optional)

Factors

1. Ask children:

To give different numerals for a number such as 12 or 36.
Teacher records responses.

12 = 8 + 4	36 = 30 + 6
12 = 4 x 3	36 = 18 + 18
12 = 24 ÷ 2	36 = 9 x 4
12 = 20 − 8 etc.	36 = 72 ÷ 2

To select those names which involve only the operation of multiplication. [12 = 4 x 3; 36 = 9 x 4]

They may rename 12 and 36 as shown below:

12 = 2 x 6	36 = 4 x 9
12 = 6 x 2	36 = 6 x 6
12 = 12 x 1	36 = 9 x 4
12 = 3 x 4 etc.	36 = 18 x 2 etc.

To record many sentences involving multiplication using the numbers below as products:

24, 21, 81, 63, 45

e.g.[24 = 3 x 8; 24 = 4 x 6; 24 = 2 x 12; 24 = 6 x 4; 24 = 24 x 1]

2. Tell children that the name given to numbers which are multiplied to obtain a product is "factors."
Record: factor x factor = product
Teacher should write the sentence: 3 x 6 = 18.
Children identify the factors. [3, 6]
Present other multiplication sentences. Ask children to identify factors.
Elicit from the children that multiplication involves finding a product when two factors are given.

3. Patterns as shown in the multiplication table.
Use rexographed outlines or graph paper to present the multiplication "table" as shown below.

X	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6							27
4	0	4								36
5										45
6										54
7										63
8										72
9										81

Include: The operational symbol "X"; 0 to 9 in the left hand column; 0 to 9 along the top row.

Have children complete one row at a time as shown above. Help them to observe patterns. Ask:

What are the intervals or the differences between two consecutive numbers in each vertical column?

What is the relationship between the intervals and the heading?

Have children examine the products in any one column.

How can you show that the product of any two factors is not affected by the order of the factors.

Note the horizontal and vertical rows of Zero.

Why are there complete lines of Zero?

What property is involved?

They note the numerals in the second row, and compare these with the heading.

Why are they the same? (Property of "1" in Multiplication)

Children note the numerals in the second column and compare these with the numerals at the left.

Why are these the same?

They should note that 12 appears four times. Why? 30 appears only twice. Why?

They find the numerals that appear only once. Why?

Multiples

1. Teacher should record as children count in sequence. (3,6,9,12,15,etc.)
   Children should be asked to count by threes through 30.
   They can list, in sequence, multiplication sentences using 3 as the constant factor. (3 = 1 x 3; 6 = 2 x 3; 9 = 3 x 3; 12 = 4 x 3; etc.)
   Each number in the sequence above is the product of two factors.
   What factor appears in each multiplication? [3]

2. Tell children that all products that have 3 as a factor are called multiples of 3. Instead of saying "3 is a factor of 12", it is often convenient to say, "12 is a multiple of 3".

3. Children should:

   List ten multiples of 3, beginning with 3. (3,6,9,12,15,etc.)

   Explain why 6 is a multiple of 3; 9 is a multiple of 3; 12 is a multiple of 3, etc.

   Name a multiple of 3.

Discuss zero as a multiple of 3, of 2, of every counting number.

Name the smallest multiple of 3; (0) the greatest (there is no greatest)

List the first 10 multiples of 2; of 3; of 4; of 6.

4. Teacher and/or children should record these sets of multiples.

   A. Multiples of 2: {0, 2, 4, 6, 8, 10, 12, ... }
   B. Multiples of 3: {0, 3, 6, 9, 12, 15, 18, ... }
   C. Multiples of 4: {0, 4, 8, 12, 16, 20, 24, ...}
   D. Multiples of 6: {0, 6, 12, 18, 24, 30, 36, ... }

   Ask the children to examine the sets above.

What can you call 12 in Set A?	[A multiple of 2]
What can you call 12 in Set B?	[A multiple of 3]
What can you call 12 in Set C?	[A multiple of 4]

   12 is a common multiple of which numbers? (a common multiple of 2, 3, 4)

   Find other multiples which appear in two or more of the sets. Explain.

5. Children should be encouraged to state the meaning of a multiple in their own words.

EVALUATION AND/OR PRACTICE

SUGGESTED EXERCISES

1. Name any multiple of 5.

2. List four multiples of 7; of 8; of 9.

3. Is 48 a multiple of 6? of 8? of 12? of 24? of 36? Justify your answers.

4. Which of the numbers below is not a multiple of 9? Why?

   83, 72, 27, 35, 62, 54, 45, 108, 225

5. Write the first twelve multiples of 10; of 100; of 1000.

6. Compare the multiples of 2 and 4.

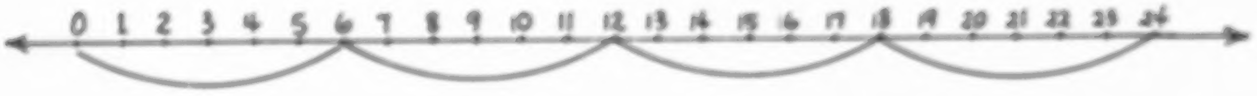
   Is every multiple of 2 also a multiple of 4? Why?

   Is every multiple of 4 also a multiple of 2? Why?

Number Line

Another approach to interpreting multiplication of whole numbers is through the use of the number line.

Present a number line:



Suggested children's activities:

Use the number line to count by threes, fours, fives, sixes, etc.

Begin with zero and state the multiples of 4, of 5, of 6, etc.

Which multiple of 6 would follow 24; 30; etc.

Facts and Extensions Beyond the Facts

1. Introduce the use of parentheses to avoid ambiguity in evaluating numerical phrases. (Order of Operations)
Record: $4 \times 6 + 2 = ?$ Discuss possible interpretations.
$(4 \times 6) + 2 = ?$ $4 \times (6 \times 2) = ?$
What should $2 + 3 \times 4$ mean? Does it mean $(2 + 3) \times 4$ or $2 + (3 \times 4)$?

Tell children that a convention used in mathematics is that in an expression such as the one above, without parentheses, multiplication should be done before addition; thus here $2 + 3 \times 4 = 14$.

Provide drill and emphasize the role of parentheses.

$(3 \times 6) + 6 = n$; $n =$

$(5 \times 6) + 18 = n$; $n =$

$12 + (3 \times 6) = n$; $n =$ etc.

Children place parentheses to make the following statements true:

$4 \times 6 + 12 = 36$; $8 \times 6 + 6 = 54$; $6 + 7 \times 6 = 48$; etc.

2. Reinforce additions to help children find products. Ability to add is essential for the application of the Distributive Property.

   Present the following additions. Follow with the multiplications that apply. For example:

   $18 + 6 = n$ For $4 \times 6 = (3 \times 6) + (1 \times 6)$
   $= 18 + 6$
   $= 24$

   ↓ ↓

   Adding 6 to multiples of 6

   Using addition to derive product. Application of Distributive Property.

   Discuss the following with children to help them see how unknown products may be derived from known products:

   $48 + 6 = n$ For $9 \times 6 = (8 \times 6) + (n \times 6) = \square$

   $300 + 6 = n$ For $51 \times 6 = (50 \times 6) + (1 \times n) = \square$

3. After drill and/or development of multiplication facts test children for automatic response to those facts. Use test results to organize children into groups for further drill.

   Suggested drills follow:

   Adding 6 to multiples of 6

   18 + 6, 48 + 6

   66 + 6, 72 + [illegible] 120 + 6, 300 + 6

   Adding multiples of 6 to multiples of 6

   18 + 12, 36 + 12, 24 + 18, 30 + 24

   60 + 12, 240 + 18, 180 + 36, 420 + 54, etc.

   Doubles of multiples of 6

   12 + 12, 18 + 18, 24 + 24

   60 + 60, 120 + 120, 240 + 240, 180 + 180, 360 + 360 etc.

Adding tens of sixes to tens of sixes

120 + 60, 180 + 60, 420 + 120, 360 + 180, etc.

4. For Multiplications

Emphasize relationships (doubling, adding groups, etc.) in deriving each product from the preceding product.

Doubling of sixes

2 x 6, 4 x 6, 8 x 6; 3 x 6, 6 x 6; etc.

8 x 6, 16 x 6, 32 x 6, etc.

6 x 6, 12 x 6, 24 x 6, etc.

30 x 6, 60 x 6, 120 x 6, etc.

Adding sixes

3 x 6, 4 x 6; 6 x 6, 7 x 6; 8 x 6, 9 x 6;

10 x 6 11 x 6; 20 x 6, 21 x 6; 30 x 6, 31 x 6; etc.

5 x 6, 7 x 6; 5 x 6, 9 x 6; 7 x 6, 9 x 6;

10 x 6, 14 x 6; 20 x 6, 27 x 6; 50 x 6, 53 x 6; etc.

Doubling and adding sixes

4 x 6, 8 x 62, 9 x 6; 3 x 6, 6 x 6, 7 x 6;

10 x 6, 20 x 6, 21 x 6; 20 x 6, 40 x 6, 43 x 6; etc.

Subtracting sixes

5 x 6, 4 x 6; 10 x 6, 9 x 6;

20 x 6, 19 x 6; 100 x 6, 90 x 6; 100 x 6, 99 x 6; etc.

Halving with sixes

100 x 6, 50 x 6; 50 x 6, 25 x 6; 25 x 6, $12 \frac{1}{2}$ x 6;

300 x 6,150 x 6; 150 x 6, 75 x 6; etc.

Using Equations

Use patterns to arrive at solutions. Ask children to explain their thinking. When children record equations, the solutions for "n" should be recorded separately, for example:

4 x 6 = 24
2 x 6 = n
n = 12

13 x 6 = 60 + n
n = 18

	From 4 x 6 = 24, derive:	From 5 x 6 = 30, derive:
(by doubling)	8 x 6 = n; n =	10 x 6 = n; n =
(by halving)	2 x 6 = n; n =	$2\frac{1}{2}$ x 6 = n; n =
(by adding sixes)	5 x 6 = n; n =	7 x 6 = n; n =
(by subtracting sixes)	3 x 6 = n; n =	4 x 6 = n; n =

From 20 x 6 = 120, derive:	From 100 x 6 = 600, derive:
40 x 6 = n; n =	200 x 6 = n; n =
10 x 6 = n; n =	50 x 6 = n; n =
21 x 6 = n; n =	130 x 6 = n; n =
19 x 6 = n; n =	99 x 6 = n; n =

Adding 1 or more sixes

5 x 6 = 30, 9 x 6 = (5 x 6) + ☐
5 x 6 = 30, 9 x 6 = 30 + (n x 6) = ☐
6 x 6 = 36, 7 x 6 = (6 x 6) + ☐
7 x 6 = 42, 9 x 6 = (7 x 6) + ☐
8 x 6 = 48, 9 x 6 = (8 x 6) + ☐
8 x 6 = 48, 9 x 6 = 48 + (n x 6) = ☐
20 x 6 = 120, 21 x 6 = (20 x 6) + ☐
20 x 6 = 120, 24 x 6 = 120 + (n x 6)
20 x 6 = 120, 27 x 6 = 120 + (n x 6)
50 x 6 = 300, 51 x 6 = (50 x 6) + ☐
50 x 6 = 300, 53 x 6 = 300 + (n x 6) = ☐

Doubling One Factor; Keeping the Other Constant

4 x 6 = 24,   8 x 6 = 24 + (n x 6);
10 x 6 = 60,   20 x 6 = (10 x 6) + (   );
40 x 6 = 240,   80 x 6 = 240 + (n x 6);
60 x 6 = 360,   120 x 6 = 360 + (n x 6) = 720;
240 x 6 = (n x 6) + 720 = □; etc.

The product of 40 x 6 is how many times as large as the product of 20 x 6? Explain. If we double one factor, what happens to the product?

How many sixes are added to 30 sixes to get 60 sixes?

Other Patterns

If 10 x 6 = 60, then 20 x 6 = (n x 6) + (n x 6)
and 30 x 6 = (□ x 6) + (n x 6)

If 5 x 6 = n, then 10 x 6 = (□ x 6) + (□ x 6)
and 15 x 6 = (□ x 6) + (? x 6)

Encourage children to find a product in a variety of ways. Record as children explain.

9 sixes = 18 + 18 + n = ?   (3 sixes + 3 sixes + 3 sixes)

9 sixes = 30 + n = ?   (5 sixes + 4 sixes)

9 sixes = 24 + 24 + n = ?   (4 sixes + 4 sixes + 1 six)

9 sixes = 60 - n = ?   (10 sixes - 1 six)

24 x 6 = 60 + 60 + n   (10 sixes + 10 sixes + 4 sixes)

24 x 6 = 72 + n   (12 sixes + 12 sixes)

24 x 6 = 120 + n   (20 sixes + 4 sixes)

123 x 6 = 600 + 120 + n   (100 sixes + 20 sixes + 3 sixes)

123 x 6 = 360 + 360 + n   (60 sixes + 60 sixes + 3 sixes)

3

$704 \times 6 = (700 \times 6) + n$

$704 \times 6 = (700 \times n) + 24$

$704 \times 6 = (700 \times n) + ?$

$704 \times 6 = (n \times 6) + 24$

$704 \times 6 = 4200 + n$

EVALUATION AND/OR PRACTICE
SUGGESTED EXERCISES

1. Write a mathematical sentence to solve the following problems. Draw an array if necessary.

   Ann has two necklaces and 4 bracelets. In how many ways can she match her necklaces and bracelets?

   Four children play the piano and 2 children play the violin. Make an array to show how many different duets can be formed consisting of a pianist with a violinist.

   How many matchings can you make of a set of 3 with a set of 6? A set of 4 with a set of 6? A set of 5 with a set of 6?

   How many dots are there in an array if the two sets being matched have 6 elements and 7 elements, respectively?

2. Interpreting Symbols; Evaluating Concepts

   Write each of the following in as many ways as you can.

   $7 + 7 + 7 + 7 = 28$

   $$\begin{array}{r} 3 \\ \underline{\times\ 5} \\ 15 \end{array}$$

   5 fours = 20

   $4 \times 6 = 24$

   Which of these can be written as multiplication facts?

   $2 + 2 + 2 + 2$ $\quad$ $3 + 3 + 3 + 4$

   $4 + 4 + 4 + 3$ $\quad$ $5 + 4 + 5 + 4$

   Write these as multiplications.

   $$\begin{array}{r} 7 \\ 7 \\ \underline{14} \end{array} \qquad \begin{array}{r} 6 \\ 12 \\ 6 \\ \underline{18} \end{array} \qquad \begin{array}{r} 8 \\ 4 \\ 4 \\ \underline{16} \end{array} \qquad \begin{array}{r} 3 \\ 3 \\ 9 \\ \underline{21} \end{array}$$

   etc.

3. One as a factor; zero as a factor

Write these as multiplication facts.

1 + 1 + 1 + 1 + 1 = 5

0 + 0 + 0 = 0

1 + 1 + 1 + 1 = 4

0 + 0 + 0 + 0 = 0

1 + 1 + 1 + 1 = □ x 1

1 + 1 + 1 + 1 + 1 + 1 = 6 x □

0 + 0 + 0 + 0 + 0 = □ x 0

0 + 0 + 0 = 3 x □

Fill in the missing numerals. Then write these as additions.

7 x 0 = ?   6 x 1 = ?

4 x 0 = ?   9 x 1 = ?

3 x 0 = ?   5 x 1 = ?

Since 6 x 1 = 6, then 1 x 6 = ?

Since 8 x 1 = 8, then 1 x 8 = ? etc.

Since 6 x 0 = 0, then 0 x 6 = ?

Since 4 x 0 = 0, then 0 x 4 = ?

4. Adding Groups

Use the same patterns and equation forms illustrated below for the more difficult facts as soon as groups of children achieve automatic response to easier facts.

Find numerals to make the following sentences true.

Adding 3's

$2 \times 3 = n + 3$
$3 \times 3 = n + 3$
$4 \times 3 = n + 3$
$5 \times 3 = n + 3$
$6 \times 3 = n + 3$

$4 \times 3 = n$
$6 \times 3 = n + 6$
$8 \times 3 = n + 6$
$10 \times 3 = n + 6$
$12 \times 3 = n + 6$

$3 \times 3 = n$
$5 \times 3 = n + 6$
$7 \times 3 = n + 6$
$9 \times 3 = n + 6$
$11 \times 3 = n + 6$ etc.

Doubling and Adding 3's

$3 \times 3 = n$
$6 \times 3 = n + 9$
$7 \times 3 = n + 3$

$3 \times 3 = n$
$6 \times 3 = n + 9$
$9 \times 3 = n + 9$
$12 \times 3 = n + 9$ etc.

Practice

$5 \times 4 = 8 + 8 + n$
$9 \times 4 = 16 + n + 4$
$7 \times 4 = n + 12 + 4$
$8 \times 4 = 24 + n$

$7 \times 5 = 15 + \square + 5$
$5 \times 5 = \square + 5$
$9 \times 5 = 15 + \square + 15$
$6 \times 5 = 25 + n$

The suggested drill for adding 10 "sixes" may be applied to any of the "tables." Continue to read as "sixes."

$10 \times 6 = n$
$20 \times 6 = n + 60 = ?$
$30 \times 6 = n + 60 = ?$
$40 \times 6 = n + 60 = ?$

$13 \times 6 = n + 18 = ?$
$23 \times 6 = n + 18 = ?$
$33 \times 6 = n + 18 = ?$
$43 \times 6 = n + 18 = ?$

$14 \times 6 = 60 + n = ?$
$24 \times 6 = 120 + n = ?$
$34 \times 6 = 180 + n = ?$
$44 \times 6 = 240 + n = ?$

$17 \times 6 = 60 + n = ?$
$27 \times 6 = 120 + n = ?$
$37 \times 6 = 180 + n = ?$
$47 \times 6 = 240 + n = ?$

Doubling One Factor

$2 \times 4 = 8$
$4 \times 4 = n$
$8 \times 4 = n$
$16 \times 4 = n$

$3 \times 4 = 12$
$6 \times 4 = n$
$12 \times 4 = n$
$24 \times 4 = n$

$7 \times 4 = 28$
$14 \times 4 = n$
$28 \times 4 = n$

$9 \times 9 = 81$
$18 \times 9 = n$
$36 \times 9 = n$

☐ x 4 = 8
☐ x 4 = 16
☐ x 4 = 32

☐ x 4 = 12
☐ x 4 = 24
☐ x 4 = 48

12 x 6 = 36 + n = ☐
14 x 6 = 42 + n = ☐
16 x 6 = 48 + n = ☐
18 x 6 = 54 + n = ☐

20 x 6 = 60 + n = ☐
40 x 6 = 120 + n = ☐
80 x 6 = 240 + n = ☐

60 x 6 = 180 + n = ☐
120 x 6 = 360 + n = ☐

Halving One Factor

4 x 8 is half as much as ? x 8
5 x 7 is half as much as ? x 7
3 x 5 is half as much as ? x 5
2 x 9 is half as much as ? x 9

5 x 3 = (10 x 3) - n = ?
5 x 5 = (10 x 5) - n = ?
5 x 6 = (10 x 6) - n = ?
5 x 4 = (10 x 4) = n = ?

☐ x 3 = 30
☐ x 3 = 15

☐ x 4 = 40
☐ x 4 = 20

☐ x 5 = 50
☐ x 5 = 25

Arriving at a Product in Various Ways

15 x 6 = 30 + 30 + n = ?	(5 sixes + 5 sixes + 5 sixes)
15 x 6 = 60 + n = ?	(10 sixes + 5 sixes)
15 x 6 = 42 + n + 6 = ?	(7 sixes + 7 sixes + 1 six)
15 x 6 = 120 - n = ?	(20 sixes - 5 sixes)

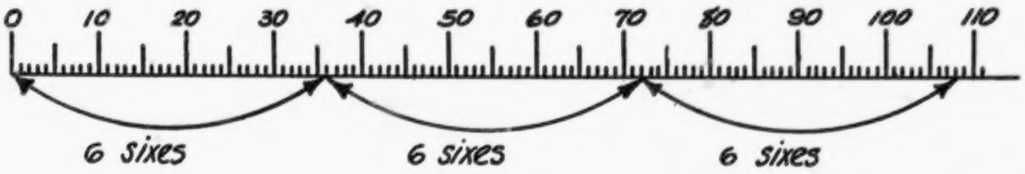
Since 7 x 6 = 42
and 14 x 6 = n
then 15 x 6 = ☐

Since 10 x 6 = 60
and 5 x 6 = n
then 15 x 6 = ☐

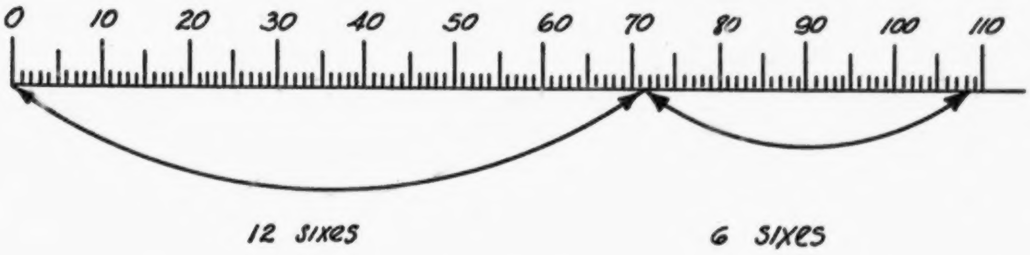
5. **Using the Number Line**

Provide children with rex[illegible]hed sheets. (5-10 number lines on each sheet) Children use [illegible]rate lines to indicate their solutions to problems, as: 1[illegible] sixes, 21 sixes, etc.

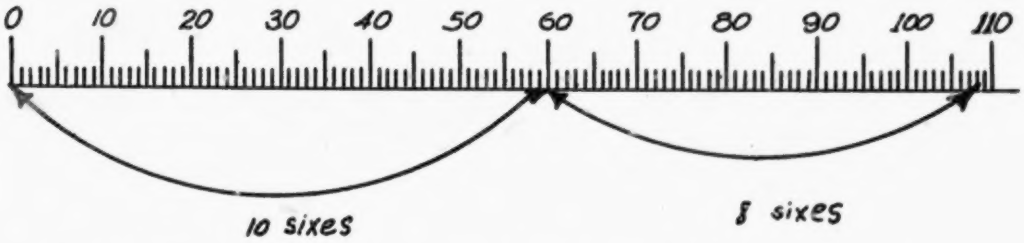
The lines below illustrate how various pupils might arrive at the product for 18 sixes.


0
10
20
30
40
50
60
70
80
90
100
110
6 sixes
6 sixes
6 sixes

6 sixes + 6 sixes + 6 sixes = 18 sixes = 36 + 36 + 36
18 sixes = 108


0
10
20
30
40
50
60
70
80
90
100
110
12 sixes
6 sixes

12 sixes + 6 sixes = 18 sixes = 72 + 36 = 108


0
10
20
30
40
50
60
70
80
90
100
110
10 sixes
8 sixes

10 sixes + 8 sixes = 18 sixes = 60 + 48 = 108

6. Write as many multiplication facts as you can to make each of these open sentences true.

☐ x Δ = 12
☐ x Δ = 18
☐ x Δ = 16
☐ x Δ = 24

☐ x ☐ = 9
☐ x ☐ = 49
☐ x ☐ = 25

7. **Fill in the missing numerals to make the open sentence true.**

$\square \times 4 = 6 \times 4 + 4$

$9 \times 3 = 6 \times 3 + \square$

$3 \times \square = 2 \times 9 + 9$

$7 \times 5 = 6 \times 5 + \square$

$3 + 3 + 3 + 3 + 3 = \square \times 3$

$7 \times 4 = 8 \times 4 - \square$

$6 \times 3 = 3 \times \square$

$\square \times 7 = 7 \times 2$

$4 \times 5 = 2 \times \square$

$8 \times \square = 4 \times 4$

$\square \times 4 = 3 \times 8$

OPERATIONS

UNIT 16 - DIVISION OF WHOLE NUMBERS: FACTS AND EXTENSIONS BEYOND THE FACTS

NOTE TO TEACHER

In division we continue to operate on two numbers to obtain another number. Division is the operation of finding one factor when the other factor and the product are known.

factor x □ = product

For example:

2 x 3 = □ requires the operation of multiplication for solution.

□ x 3 = 6
or
2 x □ = 6

require the operation of division for solution.

The relationship of the arithmetic operations is shown in the diagram below.

$$\begin{tikzcd}
\text{Addition} \arrow[r] \arrow[d, leftrightarrow] & \text{Multiplication} \arrow[d, leftrightarrow] \\
\text{Subtraction} \arrow[r] & \text{Division}
\end{tikzcd}$$

Properties of Division

Inverse Operations

"Dividing by a number" is the inverse operation of "multiplying by that number."

$$\begin{tikzcd} 4 \arrow[r, bend left, "\times 6"] \arrow[r, dash] & 24 \arrow[l, bend left, "\div 6"] \end{tikzcd}$$

$$4 \times 6 = 24$$
$$24 \div 6 = 4$$

We develop and drill division facts by relating them to associated multiplication facts.

Commutative Property

Commutativity is not a property of division.

$$8 \div 4 \neq 4 \div 8$$

Distributive Property

In division the dividend can be renamed and the Distributivity of Division with respect to Addition can be applied.

For example:

$$48 \div 8 = (40 \div 8) + (8 \div 8)$$

However the divisor cannot be renamed to apply the Distributive Property of Addition.

For example:

$$48 \div 8 \neq (48 \div 6) + (48 \div 2)$$

Closure

The set of whole numbers is not closed with respect to Division.

For example:

$8 \div 4$ is a whole number [2]

$8 \div 3$ is not a whole number

Division is not Associative

For example:

$$(8 \div 4) \div 2 = 1$$

but $8 \div (4 \div 2) = 4$

TEACHING SUGGESTIONS

Objectives: To help children understand that a division undoes its related multiplication.
To reinforce and / or develop division facts.

Procedures

Drill division facts by relating them to multiplication facts.

Facts may be recorded with words and numerals, or with numerals only. For example:

? sixes = 24 or $6\overline{)\,24}$ Children read $6\overline{)\,24}$ as "how many sixes in 24?" or "24 divided by 6".

To help children find quotients they do not know, encourage them to begin with known quotients.

To find the number of sixes in 42:
Since 6 sixes = 36, and 42 = 36 + 6,
Therefore, there are 7 sixes in 42 (1 more six)

or

Since 5 sixes = 30, and 42 = 30 + 12
Therefore, there are 7 sixes in 42 (2 more sixes)

Provide practice in adding a number of sixes to the dividend.
Adapt these suggestions for sixes to other division facts.

1. Present the following patterns and relationships.
   Have children complete the open sentences.

To find	Use Known fact	Then double
$6\overline{)\,24}$	2 x 6 = 12	☐ x 6 = 24
$6\overline{)\,48}$	4 x 6 = 24	☐ x 6 = 48
$6\overline{)\,36}$	3 x 6 = 18	☐ x 6 = 36
$6\overline{)120}$	10 x 6 = 60	☐ x 6 = 120
$6\overline{)1200}$	100 x 6 = 600	☐ x 6 = 1200

2. Using the Distributive Property

Have children find quotients:

$$6\overline{)72} \quad \text{as} \quad \begin{array}{r} n+n \\ 6\overline{)36+36} \end{array} = \square$$

$$6\overline{)96} \quad \text{as} \quad \begin{array}{r} n+n \\ 6\overline{)48+48} \end{array} = \square$$

$$6\overline{)120} \quad \text{as} \quad \begin{array}{r} n+n \\ 6\overline{)60+60} \end{array} = \square$$

Since 6 sixes = 36
Then ☐ sixes = 42 Why? [36 + 6 = 42]
And ☐ sixes = 48 Why?

Since 8 sixes = 48
Then ☐ sixes = 56 Why?

Since 10 sixes = 60
Then ☐ sixes = 66

Since 100 sixes = 600
Then ☐ sixes = 606

Adding sixes
2 sixes = 12
☐ sixes = 24 Why?
☐ sixes = 36 Why?
☐ sixes = 48 Why?

Renaming the dividend; Quotient: 10 + a set of sixes

$$\text{sixes in } 120 = 6\overline{)60} \; + \; 6\overline{)\;n} \quad \text{sixes in } 120 = 10 + \square$$

$$\text{sixes in } 78 = 6\overline{)60} \; + \; 6\overline{)\;n} \quad \text{sixes in } 78 = 10 + \square$$

$$\text{sixes in } 96 = 6\overline{)60} \; + \; 6\overline{)\;n} \quad \text{sixes in } 96 = 10 + \square$$

Mixed Patterns

$$\text{sixes in } 78 = 6\overline{)60} + 6\overline{)\;n} = \square$$

$$\text{sixes in } 78 = 10 + n = \square$$

$$\text{sixes in } 78 = 6\overline{)36} + 6\overline{)36} + 6\overline{)\;n}$$

3. **Children find the missing factor in the following sentences.**

$n \times 7 = 35$ $n = ?$ $n \times 8 = 72$ $n = ?$
$n \times 7 = 49$ $n = ?$ $n \times 8 = 80$ $n = ?$
$n \times 7 = 63$ $n = ?$ $n \times 8 = 88$ $n = ?$

4. **Test children for automatic response to division facts after drill and / or development.**

Use test results to organize children into groups for further drill.

EVALUATION and / or PRACTICE

SUGGESTED EXERCISES

1. **Answer the following:**

In the sentence $6 \times n = 42$, 6 is a known factor of the product 42.

What is "n" called? [unknown factor]
What operation can be used to find "n" [division]
What number will replace "n" to make the sentence true? [7]
Write this sentence in another form. [$6\overline{)42}$]

2. **For each multiplication fact below write a related division fact.**

$6 \times 7 = 42$ $\left[42 \div 7 = 6 \text{ or } \begin{array}{r}6\\7\overline{)42}\end{array}\right]$
$9 \times 8 = 72$ $\left[72 \div 8 = 9 \text{ or } \begin{array}{r}9\\8\overline{)72}\end{array}\right]$

3. **Rewrite each multiplication sentence as a sentence involving division.**

$6 \times 5 = n$ $[6 \times n = 30 \text{ or } 30 \div n = 6]$
$4 \times 9 = n$ $[n \times 9 = 36]$

4. Complete the following chart:

Mathematical Sentence	Operation Used	Unknown Factor
$n \times 9 = 54$	÷	?
$9 \times \square = 63$	?	7
?	÷	12

5. Write each of the following in as many ways as you can, using words.

$3\overline{)27}$ $\qquad$ $24 \div 6$

6. Rewrite these division examples using ÷ sign.

$4\overline{)20}$ $\qquad$ $9\overline{)18}$ $\qquad$ $7\overline{)56}$

At another time, the teacher might ask the children to rewrite the division example from ÷ to $\overline{)\quad}$ form.

7. Circle the numeral that shows what is to be divided. At another time, the teacher might ask the children to circle the numeral that shows the number of equivalent sets.

$8\overline{)32}$ $\qquad$ $64 \div 8 = 8$ $\qquad$ Sixes in 54 are 9

8. Write the related division fact for each of the following: (A)
Write the related multiplication fact for each of the following: (B)

A		B	
$2 \times 6 = 12$	$6\overset{2}{\overline{)12}}$	$6\overset{3}{\overline{)18}}$	$3 \times 6 = 18$
$4 \times 6 =$?	?		
$8 \times 6 =$?	?	$2\overset{?}{\overline{)12}}$	?
$9 \times 6 =$?	? etc.	$4\overset{?}{\overline{)24}}$	

9. Circle each of the following that can be divided evenly by 4.

10 11 17 14 16 19 15 8 13 6
12 18 20 31 24 29 32 28 30 35 36

10. Complete each of the following:

$4\overline{)32} = 4\overset{n}{\overline{)16}} + 4\overset{n}{\overline{)16}} = n$

$4\overline{)56} = 4\overset{n}{\overline{)28}} + 4\overset{n}{\overline{)28}} = n$

$4\overline{)72} = 4\overset{n}{\overline{)36}} + 4\overset{n}{\overline{)36}} = n$

etc.

$4\overline{)28} = 4\overset{n}{\overline{)20}} + 4\overset{n}{\overline{)8}} = n$

$4\overline{)56} = 4\overset{n}{\overline{)40}} + 4\overset{n}{\overline{)16}} = n$

$4\overline{)60} = 4\overset{n}{\overline{)40}} + 4\overset{n}{\overline{)20}} = n$

etc.

$3\overline{)24} = 3\overline{)12} + 3\overline{)n} = ?$

$4\overline{)64} = 4\overline{)32} + 4\overline{)n} = ?$

$3\overline{)54} = 3\overline{)27} + 3\overline{)n} = ?$

$3\overline{)39} = 3\overline{)30} + 3\overline{)?} = n$

$5\overline{)65} = 5\overline{)50} + 5\overline{)?} = n$

$4\overline{)92} = 4\overline{)80} + 4\overline{)?} = n$

11. Circle the dividend in each of the following:

$25 \div 5 = 5$ 35 divided by 7 equals 5 ? threes = 27

$6\overset{7}{\overline{)42}}$ There are 8 nines in 72.

Variations: Circle the divisor. Circle the quotient.

12. What is the largest remainder you can have when you divide by 5? By 4? By 9?

13. For each of the following numerals write the closest smaller number which can be divided by 7 without a remainder.

20 36 23 62 50 47 10

14. Apply these patterns to sixes, sevens, eights and nines.

Doubling the Dividend

? sevens = 14
? sevens = 14 + 14
etc.

? sevens = 28
? sevens = 28 + 28
etc.

? sevens = 21
? sevens = 21 + 21
etc.

Halving the Dividend

$7\overline{)28} = 7\overline{)14} + 7\overline{)14} = n$ $\quad$ $7\overline{)14} = 1 + n$

$7\overline{)56} = 7\overline{)28} + 7\overline{)28} = n$ $\quad$ $7\overline{)42} = 3 + n$

$7\overline{)42} = 7\overline{)21} + 7\overline{)21} = n$ $\quad$ $7\overline{)56} = 4 + n$

$7\overline{)98} = 7\overline{)49} + 7\overline{)49} = n$ $\quad$ etc.

* Enrichment Activities (Optional)

Explore some of the properties of division.

Give an example to show that division is distributive with respect to addition when the dividend is renamed.

Use the same example to show that the Property of Distribution of Division over Addition does not apply when the divisor is renamed.

Give an example to show that division is not Commutative; is not Associative; that Closure does not apply to division; that dividing by a number is the inverse operation of multiplying by that number.

SETS; NUMBER; NUMERATION

UNIT 17 - NUMBER: "BETWEENNESS"

NOTE TO TEACHER

To make reasonable estimates children should understand the range or limits within which estimates can be made.

Estimating is a form of "mental computation" to arrive at the approximate value of a mathematical computation.

To help in estimating sums, remainders, products and quotients children should understand the meaning of "betweenness", i.e., when a point on a line lies between two other points on the same line. When the points are associated with numbers on a number line children deal with both whole and fractional numbers.

Estimation involves the application of some of the properties of the fundamental operations and helps children to find answers to exercises by "mental computation".

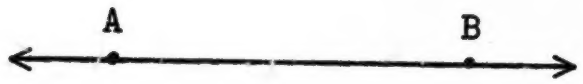
Experience situations for estimating may be found in newspapers, periodicals, tables of statistics, time lines, graphs, etc.

TEACHING SUGGESTIONS

Objective: To help children understand the concept of "betweenness" for numbers and points on a line.

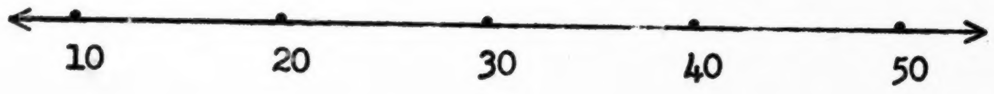
Procedures

1. Draw a line on the chalkboard. Label two points as shown.



Ask children to mark a point between A and B and label it. Discuss the position of this third point.

2. Draw a number line on the chalkboard. Mark off points and label as shown below.



Direct children to:

Name the point between 10 and 30, between 30 and 50.

Mark the point approximately midway between 30 and 40. Name it.

Mark the point approximately midway between 10 and 20. Name it.

Find a point between any points that are now on the line. Name it as accurately as you can.

3. Draw a line segment. Label points as shown.

Have children:

Mark off the mid-point between points 1 and 2 and label it.

Find a point that is midway between 1 and $1\frac{1}{2}$;

between $1\frac{1}{2}$ and 2. Label these points.

Find points midway between each of the points now named on the line segment.

Discuss the position of the points between points 1 and 2.
Ask children:

Which point is halfway between points $1\frac{1}{2}$ and 2?

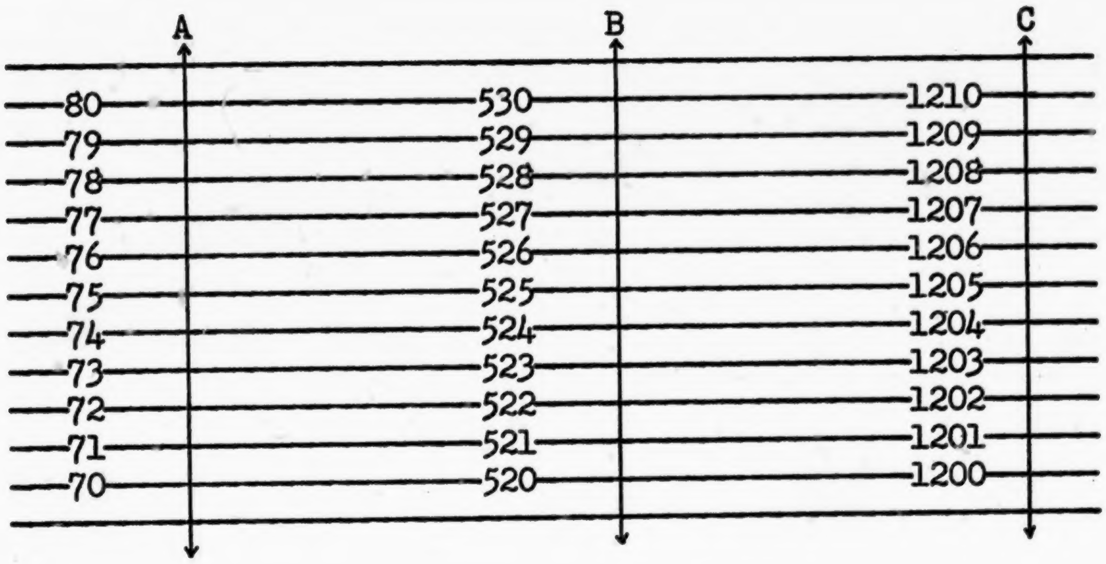
Which of these points is nearest to point 2? [$1\frac{7}{8}$]

Is $\frac{5}{8}$ nearer to point 1 or to point 2?

Is $1\frac{3}{8}$ nearer to point $1\frac{1}{2}$ or to point 2;

to point 1 or to point $1\frac{1}{2}$? ; etc.

4. Draw several vertical lines on the chalkboard. Children use lined paper. Label as illustrated.


A
B
C
80
79
78
77
76
75
74
73
72
71
70
530
529
528
527
526
525
524
523
522
521
520
1210
1209
1208
1207
1206
1205
1204
1203
1202
1201
1200

Ask children:

Which numbers are between 70 and 80?
Which number is halfway between 70 and 80?
Is 78 closer to 75 or 80? [closer to 80]
Is 75 closer to 70 or 80?

Have children note the position of each numeral on the B line in relation to 520 and 530.

Ask children:

Is 523 closer to 520 than to 530?
Is 527 closer to 530 than to 520?
Are 521, 522, 523, 524 closer to 520 than to 530?
Are 526, 527, 528, 529 closer to 530 than to 520?
Is 525 the same distance (half way) from 520 as it is from 530?

Tell children the halfway number (525) may be used as a guide in deciding whether a number is closer to next greater 10 or the next smaller 10.

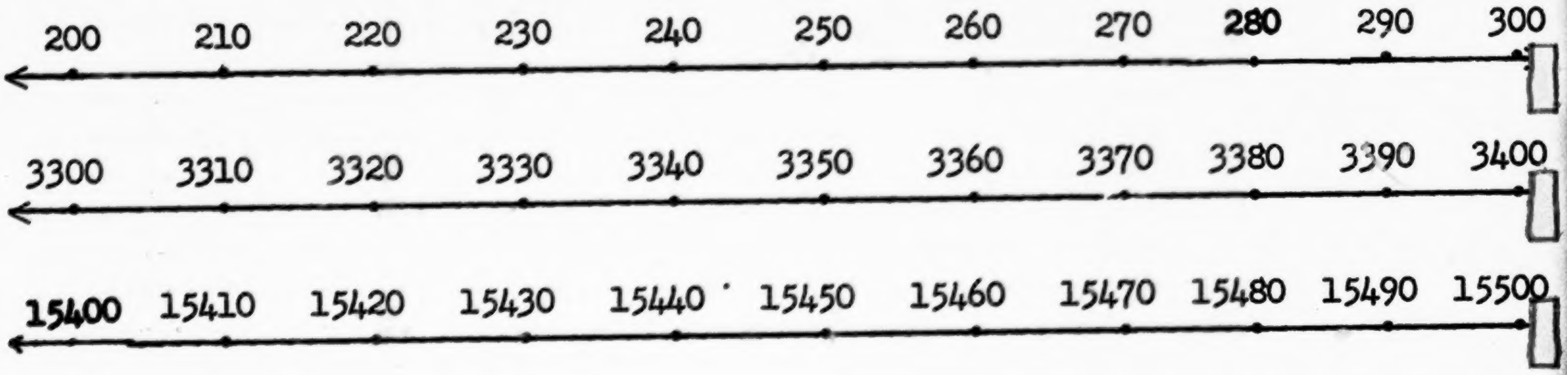
Encourage children to use a variety of terms to express their thinking, e.g.

522 is nearest in value to 520.
522 is closer to 520 than to 530.
522 is between 520 and 525.

Proceed as above for line C.

5. Draw horizontal number lines.
Children should discuss the position of each numeral on the number line in relation to:

the next higher hundred; the next lower hundred.


200
210
220
230
240
250
260
270
280
290
300
3300
3310
3320
3330
3340
3350
3360
3370
3380
3390
3400
15400
15410
15420
15430
15440
15450
15460
15470
15480
15490
15500

EVALUATION and / or PRACTICE
SUGGESTED EXERCISES

1. Which ten comes after 39? Before 41? After 99? After 89? Before 71?
Which hundred comes after 199? After 799? Before 801?
Which thousand comes after 2999? After 8999? Before 9001?

2. What is the cost of the following items to the nearest dollar?

Shoes marked $9.35 Hat marked $8.85 Dress marked $13.75

3. Give the value of each of the following to the nearest ten, the nearest hundred, the nearest thousand.

	Nearest Ten	Nearest Hundred	Nearest Thousand
5169	5170	5200	5000
3228			
12,356			
27,494			

4. 2176 is between □ thousand and Δ thousand.
It is nearest in value to ◇ thousand.

15,429 is between □ thousand and Δ thousand.
It is nearest in value to ◇ thousand.

862 is between □ hundred and Δ hundred.
It is nearest in value to ◇ hundred.

OPERATIONS

UNIT 18 - ADDITION AND SUBTRACTION OF WHOLE NUMBERS: VERTICAL FORMAT

NOTE TO TEACHER

The vertical algorithm is generally used for written computation.

An algorithm is the form used to record the steps of an arithmetic computation. It may take various forms.

At this grade level children should have the ability to rename numbers or to use expanded notation in order to deal with computations requiring exchange.

Estimating sums and remainders precedes written computation. Ways of arriving at reasonable estimates will vary depending upon the ability of the child.

Final solutions should be compared with estimates.

Verification should follow computation.

Teachers and children should be aware of how appropriate properties of operation apply to written computation.

TEACHING SUGGESTIONS

Objective: To develop skill in written computation in addition and subtraction, with emphasis on
Estimating answers
Understanding algorithms

Procedures

1. Test children's ability to rename numbers.

34 ones = ____ tens 4 ones
15 tens = ____ hundreds ____ tens
19 hundreds = ____ thousand ____ hundreds

We may exchange 93 for ____ tens 13 ones or ____ ones

We may exchange 174 for:

____ tens 4 ones
____ tens 14 ones
one hundred ____ tens 14 ones or for ____ ones

We may exchange 500 for:

____ hundreds 10 tens ____ ones
____ hundreds 9 tens ____ ones
49 tens ____ ones

Fill in the blanks:

8 dollars 6 dimes 2 pennies = ____ pennies
7 dollars 12 dimes 4 pennies = 8 dollars ____ dimes ____ pennies
4 dimes 6 pennies = 3 dimes ____ pennies
5 dimes 19 pennies = 6 dimes ____ pennies

We may exchange $6.87 for:
6 dollars 7 dimes ____ pennies
5 dollars ____ dimes 7 pennies
5 dollars 17 dimes ____ pennies
____ dimes 7 pennies

2. Reinforce children's ability to rename numbers to help them understand that renaming does not change the value of the original quantity.

Children complete the following:

3 hundreds = 300
10 tens = ☐
7 ones = 7
407

or

39 tens = ☐
17 ones = 17
☐

5 dollars = $5.00
17 dimes = ☐
17 pennies = .17
☐

5 dollars = $5.00
☐ dimes = 1.80
7 pennies = .07
$6.87

3. Emphasis should be placed on arriving at estimates of sums and remainders. The following are illustrations of various ways in which children can make estimates from less mature thinking to more mature thinking.

   Children should record estimates before they compute the exact sums and remainders.

```
 428
  89
 276
----
```

Add the hundreds. Think of 89 as about 100

Estimate: About 800 (400 + 100 + 300)

or

Add the hundreds, then the tens (600 + 170)

Estimate: About 770

or

Add the hundreds and tens as tens

(43 tens + 9 tens + 28 tens)

Estimate: About 800 (80 tens)

```
$3.12
  .89
 1.98
-----
```

Add the dollars. Think of \$.89 as about \$1 and \$1.98 as about \$2.00

Estimate: About \$6 (\$3 + \$1 + \$2)

Add the dollars, then the dimes. Think of \$.89 as about \$.90 and \$.98 as about \$1.

Estimate: About \$6 (\$4 + \$.10 + \$.90 + \$1.00)

```
 732
- 367
-----
```

Subtract the hundreds only. Estimate: about 400
Subtract 400 from 732. Estimate: 332
Subtract 36 (tens) from 73 (tens). Estimate: about 370 (37 tens)

```
 $7.25
- 3.89
------
```

Subtract the dollars only. Estimate: about \$4
Subtract \$4 from \$7. Estimate: about \$3
Subtract \$4 from \$7.25. Estimate: about \$3.25

Provide practice in estimating only, to develop facility in arriving at reasonable estimates rapidly.

Have children write their own estimate for each of the following:

$$\begin{array}{rcl} 105 + 89 + 236 & \qquad & 612 - 298 \\ 239 + 93 + 576 & & 709 - 188 \\ 72 + 134 + 681 + 9 & & 890 - 287 \\ 315 + 492 + 87 + 73 & & 596 - 129 \end{array}$$

4. Present problems. Have children estimate, compute, then verify solutions.

Children should check addition by adding in the opposite direction. (Application of the Commutative Property)

Children should check subtraction by adding the number subtracted to the remainder. (Application of the Property of Inverse Operation)

$$\begin{array}{r} 732 \\ \underline{-\ 367} \\ 365 \end{array} \qquad \text{Check:} \quad \begin{array}{rl} 365 & \text{(number left after subtraction)} \\ \underline{+367} & \text{(number that had been subtracted)} \\ 732 & \text{(number with which we started)} \end{array}$$

By subtracting the remainder from the minuend.

$$\begin{array}{r} 732 \\ \underline{-\ 367} \end{array} \qquad \text{Check:} \quad \begin{array}{r} 732 \\ \underline{-\ 365} \end{array}$$

5. Continue to develop skill in addition and subtraction.

Provide practice in addition, sums in tens, then in hundreds.
With 2-place numerals -
With 3-place numerals -
With numerals representing quantities of money - sums through $9.99

Suggested addition exercises

$$\begin{array}{rrrrrrrr} 23 & 64 & 56 & 45 & 321 & 434 & \$3.86 & \$2.35 \\ 14 & 86 & 72 & 86 & 439 & 244 & 2.98 & .47 \\ 30 & 29 & 49 & 37 & \underline{128} & 191 & \underline{1.53} & 3.21 \\ 31 & \underline{17} & \underline{37} & 59 & & \underline{36} & & \underline{1.28} \\ \underline{58} & & & \underline{73} & & & & \end{array}$$

Introduce addition with two 3-place numerals - sums in the thousands - 1 and 2 exchanges - with two numerals representing quantities of money - sums through $20 or more.

$$\begin{array}{rrrrr} 834 & 725 & 751 & \$8.89 & \$9.50 \\ \underline{449} & \underline{531} & \underline{876} & \underline{4.05} & \underline{8.75} \end{array}$$

Subtraction

Continue to develop or provide practice in subtracting from minuends through 999; from minuends through \$9.99 - 1 and 2 exchanges - 1 zero in the minuend.

$$\begin{array}{rrrrrrrrr} 80 & 56 & 387 & 464 & 807 & 520 & 148 & \$3.03 & \$8.30 \\ \underline{-\ 23} & \underline{-\ 37} & \underline{-\ 130} & \underline{-\ 356} & \underline{-\ 664} & \underline{-\ 349} & \underline{-\ \ 63} & \underline{-\ 1.25} & \underline{-5.95} \end{array}$$

Some children may be able to compute from left to right and should be encouraged to do so. Sums and remainders, therefore, are recorded from left to right. This addition may be thought through as:

$$\begin{array}{r} 23 \\ 14 \\ 30 \\ \underline{31} \end{array}$$

23, 33, 37, 67, 97, 98
Children then verify the sum by computing beginning with the ones column.

$$\begin{array}{r} 56 \\ \underline{-\ 37} \end{array}$$

This subtraction may be thought through as:
56, 26, 19

$$\begin{array}{r} 387 \\ \underline{-\ 130} \end{array}$$

This subtraction may be thought through as:
387, 287, 257

6. Be sure that children:

   Record estimates (or exact sums and remainders) before they compute with pencil and paper.

   Compare sums and remainders with estimates.
   Check sums and remainders.

   Include problem solving in the development of the addition and subtraction processes.

7. Present addition and subtraction problems using a variety of directions and indicating the processes.

   Children may find solutions to the following problems

   Find the total for 208, 89, 483, 87, 136

   Add $8 + 7 + 3 + 4 + 6 + 2 + 9 + 7 = n$

   Add

$$\begin{array}{rr} 352 & \$56.50 \\ 176 & 9.79 \\ \underline{489} & \underline{12.84} \end{array}$$

Find the sum: $\begin{array}{r} 5128 \\ \underline{3641} \end{array}$ $6.03 + $1.98 + $.72

Subtract 219 from 458 .

Find the difference between 276 and 721.

609 minus 360 = n

From 7425 subtract 5023.

Take $4.38 from $8.00

How much more than 762 is 851?

$$\begin{array}{r} \$47.79 \\ \underline{-\ \ 8.76} \end{array}$$

Underline the example you would use to find the value of "n" in
383 + n = 567

$$\begin{array}{r} 567 \\ \underline{+\ 383} \end{array} \qquad \begin{array}{r} 567 \\ \underline{\times\ 383} \end{array} \qquad \begin{array}{r} 567 \\ \underline{-\ 383} \end{array}$$

8. Include "verbal" problem solving in addition to the items suggested above.

9. Include more exercises from textbook and / or workbook.

10. Terminology

The teacher and children should use mathematical terms pertaining to addition and subtraction.

The following list suggests terms to be used rather than such expressions as "answer", "borrow", "carry", "left over", etc.

- add
- addition
- addend
- sum
- total
- plus
- subtract
- subtraction
- difference
- remainder
- minus
- place value
- estimate
- exchange
- regroup

OPERATIONS

UNIT 19 - ADDITION OF FRACTIONAL NUMBERS: COMMON FRACTIONAL FORM; HORIZONTAL FORMAT

NOTE TO TEACHER

Refer to Unit 15 for the development of concepts and comparisons for halves, fourths, eighths and thirds.

Before children can add fractional numbers they must know:

> Fractions with like denominators can be added by adding the numerators.
>
> Fractions with unlike denominators can be added by changing them to equivalent fractions with like denominators.

Addition of fractional numbers presents no problem when children understand that any fractional number may be renamed in many ways.

Encourage children to express sums of fractions with the smallest denominators. For example, $\frac{4}{8} = \frac{1}{2}$

Properties of Operations With Fractions

Properties of addition apply to fractional numbers as well as to whole numbers.

Addition of fractional numbers is Commutative
For example:

$$\frac{2}{3} + \frac{1}{3} = \frac{3}{3}$$

$$\frac{1}{3} + \frac{2}{3} = \frac{3}{3}$$

Addition of fractional numbers is Associative.
For example:

$$\frac{1}{4} + \left(\frac{3}{4} + \frac{1}{2}\right) = \left(\frac{1}{4} + \frac{3}{4}\right) + \frac{1}{2}$$

$$\frac{1}{4} + \frac{5}{4} = \frac{4}{4} + \frac{2}{4}$$

$$\frac{6}{4} = \frac{6}{4}$$

Zero is the identity element for addition of fractional numbers. For example:

$$0 + \frac{3}{4} = \frac{3}{4}$$

$$\frac{3}{4} + 0 = \frac{3}{4}$$

Objective: To develop skill in adding related fractions; halves, fourths, eighths.

TEACHING SUGGESTIONS

1. Reinforce counting forward with fractional numbers.
Use a number line and a ruler graduated in eighths.
Note: Skill in counting forward increases children's ability **to add** "mentally".

Ask children to complete the chart below.

Use number line. Fill in missing numerals.

Begin with	Size of Interval	Number of Moves	End with
0	$2\frac{1}{2}$	5	☐
2	$\frac{1}{2}$	9	☐
$3\frac{1}{2}$	$2\frac{1}{2}$	4	☐
$13\frac{1}{2}$	$1\frac{1}{4}$	☐	$19\frac{3}{4}$
☐	$\frac{1}{2}$	30	117

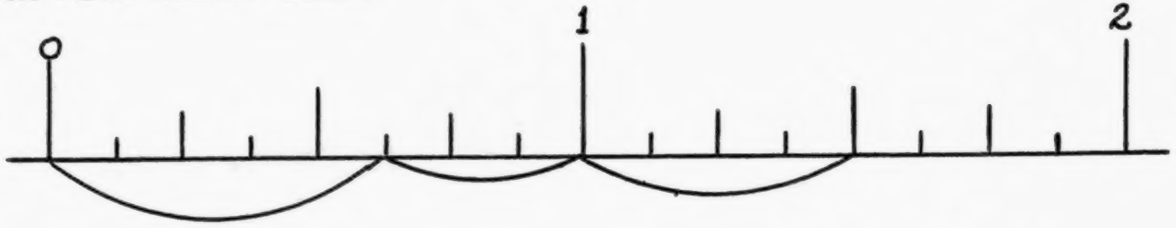
2. When adding fractions mentally children may think forward from the entire first addend and add that part of the second addend needed to reach the next whole number. (Applying The Associative Property).

Problem: We need 2 pieces of shelving paper, one $\frac{5}{8}$ yd. and the other $\frac{7}{8}$ yd. How much shelving paper shall we buy?

Children think forward to complete the whole

$$\frac{5}{8} + \frac{7}{8} = \frac{5}{8} + \frac{3}{8} + \frac{4}{8} = 1 + \frac{4}{8} = 1\frac{1}{2}$$

Check with number line.



3. Reinforce addition of fractions with like denominators. Children may use number lines and rulers where necessary.

Fraction + Fraction

$\frac{5}{8} + \frac{3}{8} = n$

$\frac{6}{8} + \frac{4}{8} = 1 + n$

$\frac{3}{4} + \frac{3}{4} = 1 + n$

$\frac{3}{2} + \frac{2}{2} = 1 + n$

Mixed Form + Whole Number

$1\frac{1}{4} + 1 = 2 + n$

$3\frac{1}{8} + 1 = 4 + n$

$4\frac{1}{2} + 3 = 7 + n$

$2\frac{1}{8} + 2 = 4 + n$

Mixed Form + Fraction

$1\frac{1}{4} + \frac{1}{4} = n$

$3\frac{1}{8} + \frac{3}{8} = n$

$4\frac{1}{2} + \frac{1}{2} = n$

$6\frac{1}{4} + \frac{2}{4} = n$

Mixed Form + Mixed Form

$1\frac{1}{4} + 1\frac{1}{4} = n$

$4\frac{1}{2} + 5\frac{1}{2} = n$

$3\frac{1}{8} + 7\frac{3}{8} = n$

$6\frac{1}{8} + 2\frac{5}{8} = n$

$3\frac{1}{8} + 2\frac{3}{8} = 5\frac{1}{8} + n = ?$

$3\frac{7}{8} + 2\frac{3}{8} = 5\frac{7}{8} + n = ?$

4. Extend to the addition of related fractions with unlike denominators.

Halves and Fourths	Halves and Eighths	Fourths and Eighths
$\frac{1}{2} + \frac{1}{4} = n$	$\frac{1}{2} + \frac{3}{8} = n$	$\frac{1}{4} + \frac{1}{8} = n$
$\frac{1}{2} + \frac{3}{4} = n$	$\frac{3}{2} + \frac{5}{8} = n$	$\frac{3}{4} + \frac{5}{8} = n$
$\frac{3}{4} + \frac{3}{2} = n$	$\frac{7}{8} + \frac{1}{8} = n$	$\frac{3}{8} + \frac{1}{4} = n$
$\frac{3}{4} + \frac{1}{2} = n$	$\frac{3}{8} + \frac{5}{2} = n$	$\frac{5}{8} + \frac{3}{4} = 1 + n$

5. Present the following exercises for practice

a. $\frac{3}{4} + \frac{1}{2} = 1 + n$

$\frac{6}{8} + \frac{1}{4} = n$

$\frac{6}{8} + \frac{3}{4} = 1 + \frac{n}{8}$

$\frac{1}{2} + \frac{5}{8} = 1 + n$

$1\frac{3}{4} + \frac{1}{2} = n$

$3\frac{3}{8} + \frac{1}{4} = n$

$2\frac{6}{8} + \frac{3}{4} = n$

$3\frac{1}{2} + \frac{3}{8} = n$

$21\frac{3}{4} + 32\frac{1}{2} = n$

$13\frac{3}{8} + 23\frac{1}{4} = n$

$52\frac{6}{8} \div 41\frac{3}{4} = n$

$2\frac{7}{8} + 1 = n$

$2\frac{7}{8} + 1\frac{3}{4} = n$

$2\frac{7}{8} + 1\frac{3}{4} = 3\frac{7}{8} \div n = ?$

b. Add-Use number line where necessary.

$2\frac{1}{2} + 2\frac{1}{2} + 2\frac{1}{2} = n$

$1\frac{1}{4} + 2\frac{1}{2} + 3\frac{3}{4} = n$

$32\frac{7}{8} + \frac{3}{4} + 3\frac{1}{2} = n$

$82\frac{1}{8} + 1\frac{3}{4} + 2\frac{1}{2} + 3\frac{3}{8} = n$

c. Supply the missing numerals:

$16 + \square + \square = 23$

$11 = \square + \square$

$\frac{\square}{4} + \frac{\square}{4} = 10$

$\frac{\square}{8} + \frac{\Delta}{8} = 10$

6. Children complete the following. They **should explain the property involved.**

$\frac{3}{4} + \frac{3}{8} = \frac{3}{4} + \left(\frac{\square}{8} + \frac{1}{8}\right)$

$\frac{3}{4} + \frac{1}{2} = \frac{1}{2} + \frac{3}{\square}$

$5\frac{3}{4} + \frac{5}{8} = 5 + \left(\frac{\square}{8} + \frac{5}{8}\right)$

$0 + \frac{2}{3} = \frac{2}{3} + \square = n$

$\frac{3}{8} + \square = \frac{3}{8}$

$\frac{7}{8} + \frac{1}{4} = \frac{1}{4} + \square$

OPERATIONS

UNIT 20 - SET OF FRACTIONAL NUMBERS: SUBTRACTION; COMMON FRACTIONAL FORM; HORIZONTAL FORMAT

NOTE TO TEACHER

When fractional numbers are subtracted the denominators, just as in addition, must be the same.

Since we are dealing with a set of positive numbers, the first fraction must be greater than the second, when subtracting.

For example: Since $\frac{3}{4} > \frac{1}{4}$ we may subtract $\frac{1}{4}$ from $\frac{3}{4}$ and have a result which is a positive number.

Subtraction of fractional numbers may also be thought of as finding a missing addend.

For example:

Addition: Addend + addend = □

$$\frac{3}{4} + \frac{1}{4} = \square$$

$$\text{Subtraction} \begin{cases} \frac{3}{4} + \square = \frac{4}{4} \\ \square + \frac{1}{4} = \frac{4}{4} \end{cases} \qquad \begin{array}{rl} \frac{4}{4} & \text{(Sum)} \\ -\frac{1}{4} & \text{(Addend)} \\ \hline \frac{3}{4} & \text{(Addend)} \end{array}$$

Subtraction: Addend + □ = Sum

Subtracting a number is the inverse operation of adding that number. Therefore the solution to a subtraction problem involving fractional numbers may be verified by applying the inverse operation, addition.

Equivalent Fractions

Equivalent fractions are fractions that name the same fractional number. For example:

$$\frac{1}{2} = \frac{2}{4}, \qquad \frac{1}{2} = \frac{3}{6}$$

The renaming of fractions to their equivalents has been taught to children in previous grades through the use of experience materials and representative materials. In a later Unit of Grade 5 the Fundamental Property of Fractions will be developed.

The Fundamental Property of Fractions: If the numerator or denominator of a fraction is multiplied or divided by the same non-zero number, the resulting fraction is equivalent to the original fraction.

$$\frac{a}{b} = \frac{a \times c}{b \times c}, \qquad c \neq 0$$

Ability to find equivalents for fractional numbers (renaming fractions) is essential.

For example:

To subtract $8\frac{3}{4} - 3\frac{7}{8}$ it is essential that children know that $\frac{3}{4}$ must first be changed to the equivalent fraction $\frac{6}{8}$.

At this time children can solve $8\frac{6}{8} - 3\frac{7}{8} = n$ by applying the Associative Property for Addition.

$3\frac{7}{8} = 3 + \frac{7}{8}$ which will be thought of as:

$3 + \left(\frac{6}{8} + \frac{1}{8}\right).$

Then to solve $8\frac{6}{8} - 3\frac{7}{8} = n$ children can think:

$5\frac{6}{8} - \frac{7}{8} = n$ (after subtracting 3 from $8\frac{6}{8}$)

$5\frac{6}{8} - \frac{6}{8} + \frac{1}{8} = n$ (after renaming $\frac{7}{8}$ as $\frac{6}{8} + \frac{1}{8}$)

$5 - \frac{1}{8} = n$ (after subtracting $\frac{6}{8}$ from $5\frac{6}{8}$)

$4\frac{8}{8} - \frac{1}{8} = n$ (after renaming 5 as $4\frac{8}{8}$)

$4\frac{8}{8} - \frac{1}{8} = 4\frac{7}{8}$ (solution, $4\frac{7}{8}$)

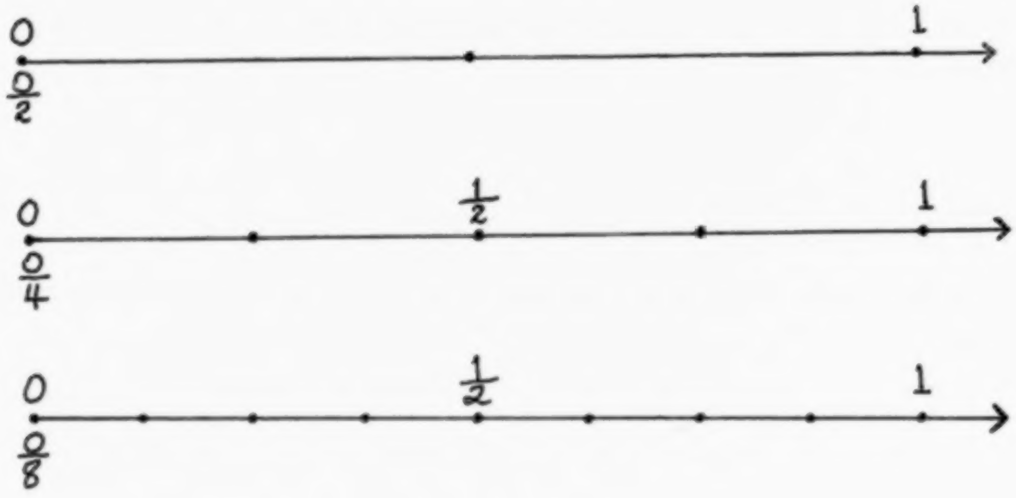
Objectives: To reinforce subtraction of fractional numbers involving halves, fourths, eighths

Meaning of equivalent fractions
Renaming numbers - Equivalents
Inequalities
Counting
Finding the missing addend

TEACHING SUGGESTIONS

Equivalents

1. Use number rays to reinforce equivalents.



Have children:

Assign numbers to the points on the rays above
Assign another number name to the point named by "1" on each of the rays to correspond with the way the ray has been divided. Record the new name below the point. $\left[\frac{2}{2}, \frac{4}{4}, \frac{8}{8}\right]$

Assign another name to the point named by " $\frac{1}{2}$ " on each of the rays.

Record the new name below the point.

Children record the set of equivalent fractions for $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$

$$\left[\frac{1}{2}, \frac{2}{4}, \frac{4}{8}, \text{etc.}\right]$$

2. Discuss the notations $\frac{0}{2}$, $\frac{0}{4}$, $\frac{0}{8}$. To what whole number is each equivalent? [Zero]

3. Children:

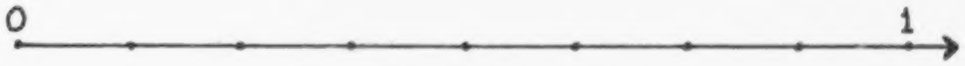
   Record another name for $\frac{8}{8}$, $\frac{4}{4}$, $\frac{2}{2}$

   Explain $\frac{1}{1} = 1$

4. Extend number rays to include more than one unit and follow same procedure.

Inequalities

Present a number ray. Divide one unit into halves, then fourths, then eighths.



Have children find the solution sets for the following:

The points show that the unit on the ray is divided into ____ parts.

a. $\square > \frac{1}{2}$

b. $\frac{3}{8} < \square$

c. $\frac{\square}{8} > \frac{3}{4}$

d. $1\frac{1}{2} \ \square \ \frac{8}{8}$

a. $\frac{5}{8}$, $\frac{6}{8}$, $\frac{7}{8}$, 1

b. $\frac{4}{8}$, $\frac{5}{8}$, $\frac{6}{8}$, $\frac{7}{8}$, 1

c. $\frac{7}{8}$, $\frac{8}{8}$,

d. >

Counting

1. Reinforce counting backward by subtracting various fractional numbers. Use number lines.

Direct children to count backward from:

$4\frac{1}{2}$, by halves to 2

$5\frac{1}{8}$, by eighths to 2

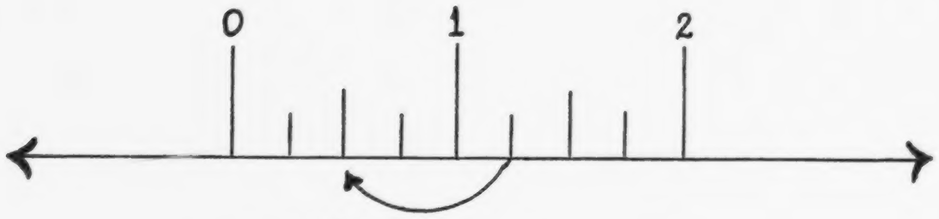
$2\frac{1}{4}$, by eighths to 1

$5\frac{3}{4}$, by $\frac{3}{4}$ to $4\frac{1}{2}$

Children explain in their own words how they know they have counted by halves, eighths, etc.

2. Present a subtraction problem involving fractional numbers. Children should solve it by counting backward on a number line.

Problem: The recipe for a pie calls for $1\frac{1}{4}$ cups of flour. It tells us to use $\frac{3}{4}$ cup for the crust. How much is left for filling?



$1\frac{1}{4} - \frac{3}{4} = n$

3. Ask children to evaluate each of the following mentally: Check some using the number line.

$2\frac{1}{4} - 1 = n$ $1\frac{7}{8} - \frac{3}{8} = n$ $5\frac{3}{4} - 1\frac{3}{4} = n$
$4\frac{1}{2} - 2 = n$ $9\frac{5}{8} - \frac{2}{8} = n$ $7\frac{5}{8} - 3\frac{5}{8} = n$
$5\frac{1}{8} - 2 = n$ $5\frac{3}{4} - \frac{1}{4} = n$ $4\frac{1}{2} - 1\frac{1}{2} = n$
$5\frac{3}{4} - 3 = n$ $6\frac{3}{4} - \frac{3}{4} = n$ $6\frac{7}{8} - 4\frac{7}{8} = n$

$5\frac{3}{4} - 1\frac{1}{4} = n$ $2\frac{1}{8} - 1 = n$ $3\frac{1}{3} - 1\frac{2}{3} = n$
$6\frac{7}{8} - 5\frac{5}{8} = n$ $2\frac{1}{8} - \frac{1}{8} = n$ $4\frac{1}{4} - 2\frac{3}{4} = n$
$9\frac{5}{8} - 3\frac{2}{8} = n$ $2\frac{1}{8} - 1\frac{1}{8} = n$ $8\frac{3}{8} - 3\frac{7}{8} = n$
$4\frac{2}{4} - 1\frac{1}{4} = n$ $2\frac{1}{8} - \frac{2}{8} = n$ $4\frac{1}{8} - 2\frac{5}{8} = n$

Applying the Associative Property for Addition

1. Reinforce renaming fractional numbers and mixed forms to prepare for

applying the Associative Property for Addition. For example:

Rename $9\frac{6}{8}$ means $9 + \square$

$9\frac{6}{8} = 8 + \frac{\square}{8}$

Rename $17\frac{3}{4}$ means $17 + \square$

$17\frac{3}{4} = 16 + \frac{\square}{4}$

2. Solve for n in the following exercises. Use the Associative Property of Addition where necessary.

$1\frac{5}{8} - \frac{1}{4} = n$

$\frac{7}{8} - \frac{1}{2} = n$

$\frac{5}{8} - \frac{1}{2} = n$

$38\frac{7}{8} - 23\frac{3}{4} = n$

$17\frac{5}{8} - 12\frac{3}{4} = n$

$8\frac{1}{2} - 4\frac{3}{4} = n$

$4\frac{3}{8} - 2\frac{1}{2} = n$

Finding the Missing Addend

1. Find the missing addend:

$37 + \square = 50$ $129 = \square + 109$ $3415 + n = 3525$

Ask the following questions:

What is the sum in each of the problems above?

What are the other numbers in the equation called?

[addend]

What operation can you use to find the missing addend in each case?

[subtraction]

2. Present an exercise such as: $8\frac{3}{4} + 5\frac{1}{8} = \square$

Ask children to:

Find the sum
Name the addends

3. Present the exercise in a word problem: $9\frac{1}{4} + \square = 16\frac{5}{8}$

Children:

Make the open sentence true.

Tell how they found the missing addend.

Record the equation to show subtraction. $\left[16\frac{5}{8} - 7\frac{3}{8} = 9\frac{1}{4}\right]$

Name the addends in the subtraction above. $\left[7\frac{3}{8}\,,\ 9\frac{1}{4}\right]$

Name the sum. $\left[16\frac{5}{8}\right]$

Write each of the subtraction exercises below as an addition exercise:

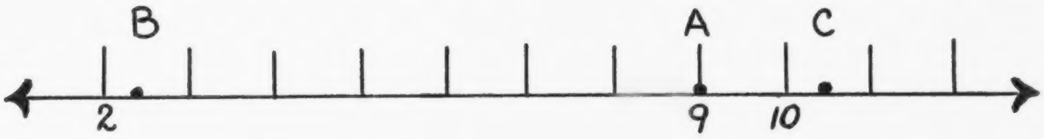
$4\frac{5}{8} - 1\frac{3}{8} = 3\frac{1}{4}$ $\left[4\frac{5}{8} = 3\frac{1}{4} + 1\frac{3}{8}\right]$

$1\frac{7}{8} - \frac{3}{2} = \frac{3}{8}$ $\left[\frac{3}{8} + \frac{3}{2} = 1\frac{7}{8}\right]$

4. Discuss the following. Children find solutions.

We have three points, A, B, C, on a line.
Which two of the three points are farthest apart?
How far apart?

A. 9 B. $2\frac{1}{2}$ C. $10\frac{1}{2}$



Assign other numbers to points A, B, C. Proceed as above.

A.	B.	C.
$10\frac{1}{2}$	$6\frac{1}{2}$	$8\frac{1}{2}$
$\frac{8}{4}$	$\frac{9}{4}$	$\frac{10}{4}$
$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$
5	$5\frac{3}{8}$	$5\frac{3}{4}$

5. Additional practice exercises

$5\frac{1}{4} + \square = 5\frac{3}{8}$

$5\frac{1}{4} + \square = 14$

$5\frac{1}{4} + \square = 11\frac{1}{2}$

$5\frac{1}{4} + 1\frac{\square}{\triangle} = 7\frac{1}{8}$

$7\frac{5}{8} - 2\frac{3}{4} = 5\frac{5}{8} - n$

$8\frac{1}{2} - 4\frac{3}{4} = 4\frac{1}{2} - n$

$5\frac{3}{4} - 3\frac{7}{8} = 2 \quad - n$

GEOMETRY AND MEASUREMENT

UNIT 21 - MEASUREMENT: CAPACITY; CONSERVATION; EQUIVALENTS

NOTE TO TEACHER

In Grade 5 continue to emphasize:
A Principle of Conservation in Science
That all Measurements are approximate.

In science a Principle of Conservation refers to the fact that certain properties of things do not change even when conditions about them change. For example, one quart is always one quart whether the quart is distributed in a quart jar or in a gallon jar; one quart is one quart whether it is distributed in a quart milk bottle or in a mayonnaise jar of a one quart capacity.

Understanding conservation may help children to understand equivalents among measures.

Children should also understand that two different substances may have the same weight even though they have different volumes. A pound of feathers occupies more space than a pound of iron **but both weigh one pound.**

Objectives: To reinforce concepts of conservation.
To organize relationships among measurements.
To help children understand capacity.

TEACHING SUGGESTIONS

1. Use classroom experiences such as:

Preparing for Health Day
Discuss and plan healthful menus
Discuss amounts of basic foods needed in daily diet

Planning for a Halloween Party

Measure ingredients when preparing food
Estimate quantity of juice or cider needed - using a cup, glass, pint or quart container
(32 cups, or 8 quarts, or 2 gallons, or 4 half gallons)
Double or triple the amount given in recipes

2. Reinforce equivalents among the number of cups, pints, quarts and gallons. (e.g. 2 pints = 1 quart, etc.)

3. Introduce the fluid ounce

Materials needed:

Standard measuring spoons
1 ounce glass
Perfume bottle
Water

Children compare capacities in fluid ounces using measuring spoon; in fluid ounces with standard cup.

Discuss uses of fluid ounce.
Discuss the principle of conservation

Note the need for a unit of measure between the spoon and the cup.

4. Provide practice in changing from larger units to smaller units; smaller units to larger units.

For example:
$2\frac{1}{2}$ quarts = 2 quarts and 1 pint, or 5 pints, or 10 cups
12 pints = 6 quarts, or 1 gallon and 2 quarts, or $1\frac{1}{2}$ gallons,
or 1 gallon and 4 pints

Since 8 pints = 1 gallon, then 1 pint = ? gallon

Children should note that when changing from smaller units to larger units the number of units is fewer; when changing from larger units to smaller units the number of units is greater. Try to have students discover this and state generalization.

5. Have children organize relationships among measurements including fractional units.

2 cups = 1 pint   1 cup = $\frac{1}{2}$ pint
2 pints = 1 quart   1 pint = $\frac{1}{2}$ quart
4 quarts = 1 gallon   1 quart = $\frac{1}{4}$ gallon
8 pints = 1 gallon   1 pint = $\frac{1}{8}$ gallon

6. Introduce concepts of dry measure.

Discuss:

Need for containers for dry measure.
Products sold in units of dry measure.
[berries, apples, vegetables, etc.]

Similarity of names for some units of dry measure and some units of liquid measure.
[pints, quarts]

7. Have children explore different units used on products found in supermarkets; in books; etc.

8. Introduce terms: peck, bushel.

Have children bring in containers, such as bushel, etc.

9. Use transparent containers of the same size.
Fill one container with beads that are 1" in diameter, the other with beads that are $\frac{1}{2}$" in diameter.

Children note that the number of objects and the weight of objects may vary although the capacity is the same.

10. Children organize "table" of dry measure:

2 pints = 1 quart
8 quarts = 1 peck (pk.)
4 pecks = 1 bushel (bu.)

EVALUATION and / or PRACTICE
SUGGESTED EXERCISES

1. Choose the correct unit of measure.

    An oil dealer delivers oil in

    pints, quarts, gallons

2. Name two liquid products measured in quarts;
    two dry products measured in quarts.

3. Are there more green peas or more potatoes in a bushel? Why?

4. Solve the following equations:

    2 quarts = □ gallon

    2 pints = □ gallon

    2 gallons = □ quart

5. Additional practice exercises may be found in textbooks.

OPERATIONS

UNIT 22 - SET OF WHOLE NUMBERS: MULTIPLICATION; HORIZONTAL FORMAT

NOTE TO TEACHER

To multiply "mentally", children must have the following background:

Ability to rename numbers using expanded notation, for example,

79 = 70 + 9; 246 = 200 + 40 + 6

Ability to add "mentally" (left to right computation) numbers through 999, for example, 320 + 81 = ☐

Ability to multiply "mentally" whole decade and whole hundred numbers with one factor through 9. For example,
9 x 30 = 270; 9 x 300 = 2700.

Automatic response to multiplication facts.

Steps in developing understanding and skill in multiplying "mentally" are:

Teacher should present a problem situation familiar to the children.

Children then should determine that the problem calls for multiplication.

Teacher records the problem as a mathematical sentence.

Children then compute "mentally" from left to right.

They should arrive at products in a variety of ways.

They record only the products.
Teacher records their thinking on the board. For example:

For 4 x 68 = ☐

Child May Think	Teacher Should Write on Chalkboard
4 sixties are 240	(4 x 60) + (4 x 8) = n
4 eights are 32	240 + 32 = 272
240 and 32 are 272	

(Regrouping one factor and applying the Distributive Property)

or

Child May Think	Teacher Should Write on Chalkboard
2 sixty-eights are 136	(2 x 68) + (2 x 68) = n
2 more sixty-eights are 136	136 + 136 = 272
136 and 136 equal 272	

(Regrouping the other factor)

For 3 x 258 = ☐

Child May Think	Teacher Should Write on Chalkboard
3 two hundreds are 600	3 x 258 = n
3 fifties are 150	600 + 150 + 24 = 750 + 24
600 and 150 are 750 and	= 774
24 more are 774	

or

2 two hundred fifties are 500	3 x 258 = 500 + 250 + 24
and another 250 would be 750	= 774
Add 24 more and that would be 774	

For 2 x 637 = ☐

Child May Think	Teacher Should Write on Chalkboard
2 six hundreds are	2 x 637 = 1200 + 74 = 1274
12 hundred and 74 more	
(2 thirty-sevens)	
equals 1274	

Objective: To develop skill in mental multiplication by a one-digit factor. To develop understanding why the horizontal algorithm works as an application of the Distributive property.

TEACHING SUGGESTIONS

The teacher should present exercises or problems whose numbers are within the capabilities of the children.

One Factor a Two Digit Numeral; 2 and 3 Digit Numerals in the Product;

For example, 6 x 27 = □ 4 x 324 = □

Before multiplying whole decade numbers, practice should be given in adding, in sequence and out of sequence with whole decade numbers.

For example:

1. Additions

Add 40 to successive multiples of 40; 40 + 40 = □ 80 + 80 = □, etc.
Add 80 to successive multiples of 40; 40 + 80 = □ 120 + 80 = □, etc.
Add 80 to successive multiples of 80; 80 + 80 = □ 160 + 80 = □, etc.

Double 40 and its multiples in and out of sequence.

40 + 40 = □ 80 + 80 = □ 40 + 40 = □ 160 + 40 = □, etc.

2. Multiplications

Multiplying in Sequence 2 x 40 = □ 3 x 40 = □ 4 x 40 = □ etc.

Doubling One Factor 2 x 40 = □ 4 x 40 = □ 8 x 40 = □ etc.
3 x 40 = □ 6 x 40 = □

Doubling and Adding 40's 3 x 40 = □ 6 x 40 = □ 7 x 40 = □
3 x 40 = □ 6 x 40 = □ 8 x 40 = □

From time to time record the children's thinking.

For 8 x 40 record:

(4 x 40) + (4 x 40) = 160 + 160 = 320
or
(5 x 40) + (3 x 40) = 200 + 120 = 320
or
(3 x 40) + (3 x 40) + (2 x 40) = 120 + 120 + 80 = 320

3. Extend to multiplications in which one factor is a 2 digit number.

Present one set of exercises at a time

Illustrative series:

Since $3 \times 60 = 180$
then $3 \times 67 = n$

Since $4 \times 90 = 360$
then $4 \times 96 = n$

$2 \times 63 = \square + 6 = n$
$4 \times 63 = \square + 12 = n$
$8 \times 63 = \square + 24 = n$

$3 \times 63 = \square + 9 = ?$
$6 \times 63 = \square + 18 = ?$

Since $2 \times 80 = 160$
then $4 \times 80 = n$
$4 \times 86 = n$

Since $3 \times 60 = 180$
then $6 \times 60 = n$
$6 \times 62 = n$

$2 \times 24 = \square$
$4 \times 24 = \square + \square = ?$

$2 \times 37 = \square$
$4 \times 37 = \square + \square = ?$ etc.

$3 \times 21 = \square$
$6 \times 21 = \square + \square = ?$

$3 \times 54 = \square$
$6 \times 54 = \square + \square = ?$

A One Digit Factor x a Three Digit Factor

Suggested Types of Exercises

Children add whole hundreds before multiplying with whole hundreds.

Additions

Add 400 to successive multiples of 400;
$400 + 400 = \square$ $800 + 400 = \square$ etc.

Add 400 to multiples of 400 out of sequence:
$1200 + 400 = \square$ $2800 + 400 = \square$ etc.

Add 800 to multiples of 400 out of sequence:
$1200 + 800 = \square$ $2800 + 800 = \square$ etc.

Double 400 and its multiples in and out of sequence:
$400 + 400 = \square$ $800 + 800 = \square$ etc.

2. Multiplications

Multiplying in Sequence: $2 \times 400 = \square$ $3 \times 400 = \square$ $4 \times 400 = \square$

Doubling the Multiplier: 2 x 400 = □ 4 x 400 = □ 8 x 400 = □
3 x 400 = □ 6 x 400 = □

Doubling and Adding Sets of 400
3 x 400 = □ 6 x 400 = □ 7 x 400 = □
3 x 400 = □ 6 x 400 = □ 8 x 400 = □

From time to time record the children's thinking.

Use Equations - Apply Distributive Property with respect to Addition.

Adding in Sequence

2 x 400 = □ + 400
3 x 400 = □ + 400
4 x 400 = □ + 400

Doubling

2 x 400 = □
4 x 400 = □ + □
8 x 400 = □ + □

3 x 400 = □
6 x 400 = □ + □

Doubling and Adding Groups

2 x 400 = □
4 x 400 = □ + □ = n
5 x 400 = □ + 400 = n

3 x 400 = 1200
6 x 400 = □ + □ = n
7 x 400 = □ + 400 = n

4 x 400 = 1600
8 x 400 = □ + □ = n
9 x 400 = □ + 400 = n

3 x 400 = □
6 x 400 = □ + □ = n
9 x 400 = □ + 1200 = n

3. Extend to multiplication with more difficult numbers.

Adding Groups

4 x 200 = 800
4 x 223 = 800 + n = n
4 x 254 = 800 + n = n
4 x 285 = 800 + n = n

2 x 839 = n + 78 = n
5 x 212 = n + 60 = n
4 x 342 = n + 168 = n
8 x 105 = n + 40 = n
3 x 521 = n + 63 = n

Doubling One Factor

2 x 316 = □ + 32 = n
4 x 316 = □ + 64 = n
8 x 316 = □ + 128 = n

3 x 316 = □ + 48 = n
6 x 316 = □ + 96 = n

Varied Patterns Using Same Factor

2 x 749 = 1400 + n = □
2 x 749 = n + 18 = □
2 x 749 = n + 98 = □
2 x 749 = 1500 - n = □

$4 \times 243 = 800 + n + 12 = \square$
$4 \times 243 = 486 + n = \square$
$4 \times 243 = n + 12 = \square$

Present one set of equations at a time. Illustrative series:

Since $2 \times 400 = 800$
Then $2 \times 448 = n$

Since $2 \times 400 = 800$
Then $2 \times 484 = n$

Since $2 \times 400 = 800$
Then $4 \times 400 = n$
$4 \times 405 = n$

Since $3 \times 400 = 1200$
Then $6 \times 400 = n$
$6 \times 411 = n$

Since $4 \times 400 = 1600$
Then $8 \times 400 = n$
$8 \times 402 = n$

Since $3 \times 400 = 1200$
Then $3 \times 425 = n$

Since $4 \times 400 = 1600$
Then $4 \times 408 = n$

EVALUATION and / or PRACTICE

SUGGESTED EXERCISES

Solve the following equations:

1. **Adding in Sequence**

$2 \times 30 = \square + 30$
$3 \times 30 = \square + 30$
$4 \times 30 = \square + 30$
through
$9 \times 30 = \square + 30$

Doubling and Adding Groups

$2 \times 30 = \square$
$4 \times 30 = \square + \square = ?$
$8 \times 30 = \square + \square = ?$
$9 \times 30 = \square + 30$

Doubling Groups

$2 \times 30 = \square$
$4 \times 30 = \square + \square$
$8 \times 30 = \square + \square$
$3 \times 30 = \square$
$6 \times 30 = \square + \square$

$3 \times 30 = 90$
$6 \times 30 = \square + \square = ?$
$7 \times 30 = \square + 30$

2. Find the missing number (□) and then solve for "n".

2 x 32 = □ + 4 = n
2 x 69 = □ + 18 = n
2 x 57 = □ + 14 = n

3 x 23 = □ + 9 = n
3 x 58 = □ + 24 = n
3 x 34 = □ + 12 = n
3 x 62 = □ + 6 = n etc.

2 x 37 = 60 + □ = n
2 x 27 = 40 + □ = n
2 x 78 = 140 + □ = n

3 x 43 = 120 + □ = n
3 x 36 = 90 + □ = n
3 x 64 = 180 + □ = n
3 x 59 = 150 + □ = n etc.

Increase the Difficulty

4 x 36 = □ + 24 = n
3 x 52 = □ + 6 = n
5 x 29 = □ + 45 = n
4 x 83 = □ + 12 = n

2 x 37 = □ + 14 = n
3 x 37 = □ + 21 = n
4 x 37 = □ + 28 = n
5 x 37 = □ + 35 = n - through -
9 x 37 = □ + 63 = n

3. Since 2 x 300 = 600 then
2 x 309 = 600 + n
2 x 330 = 600 + n
2 x 345 = 600 + n
2 x 360 = 600 + n
2 x 375 = 600 + n

Since 2 x 300 = 600 then
2 x 321 = 600 + n
2 x 342 = 600 + n
2 x 384 = 600 + n

Since 4 x 300 = 1200 then
4 x 304 = ?
4 x 311 = ?
4 x 325 = ?

Since 4 x 300 = 1200 then
4 x 309 = 1200 + n
4 x 318 = 1200 + n
4 x 336 = 1200 + n

4. Evaluate the following exercises. Tell how they are alike. How are they different? Explain the property or properties involved.

(5 x 4) x 3 = n
5 x (4 x 3) = n
(4 x 5) x 3 = n
4 x (5 x 3) = n
(3 x 4) x 5 = n
3 x (4 x 5) = n

OPERATIONS

UNIT 23 - SET OF WHOLE NUMBERS: MULTIPLICATION; VERTICAL FORMAT

NOTE TO TEACHER

The same properties of the operation of multiplication which children have used when multiplying using a horizontal format algorithm are also used when using a vertical format algorithm.

The vertical algorithm for multiplication depends especially upon the application of the Distributive Property of Multiplication with respect to Addition.

Children may use a variety of algorithms in computing products.

The numbers children should be given will be determined by:

1. Ability to multiply in horizontal format.
2. Understanding of Place Value
3. Ability to express numbers in expanded notation
4. Understanding of the Distributive Property

Some children should be given more difficult computations such as:

5 x 89, 7 x 256, 6 x 629, 8 x 609, etc.;

Others, less mature, simpler computations such as:

6 x 24, 5 x 142, 2 x 697, 3 x 307, etc.

Multiplication situations should be presented in different ways.

For example:

5 x 807 = n
Multiply 5 by 807
Multiply 5 and 807

The factors are 5 and 807. Find the product.
Find the product of 5 and 807.
Find the value of 807 nickels.

$$\begin{array}{r} 807 \\ \underline{\times\ 5} \end{array}$$

Problems may be read in a variety of ways.

Problem	May Be Read As:
5 x 847 = n	5 eight hundred forty sevens
$\begin{array}{r} 847 \\ \underline{\times\ 5} \end{array}$	5 times 847
	847 taken 5 times
	847 multiplied by 5

Numerals may be read in various ways.
Children should decide on the most convenient way for a specific purpose

4235 may be read as:
Forty two hundred thirty five

or as:
4 thousand 2 hundred thirty five.

Products may be estimated in a variety of ways.

Problem: 5 x 847 = n

Children may think:

5 x 800 = 4000　　5 x 800 = 4000
5 x 40 = 200　　5 x 50 = 250

Children record estimate

5 x 847 > 4200; or 5 x 847 < 4250;
or, the product of 5 x 847 is between 4200 and 4250.

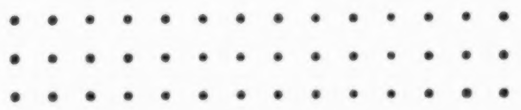
TEACHING SUGGESTIONS

Objective: To develop skill in the techniques of multiplication, using the vertical format.

1. Reinforce understanding of the meaning and method of the vertical algorithm using arrays.

Present an exercise: $3 \times 14 = n$

Have children:
Draw an array to represent the multiplication.

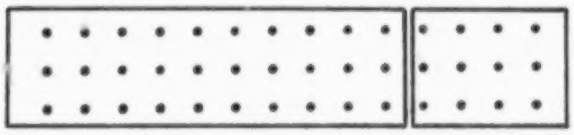


Express 14 in expanded notation. $[\ 14 = (\ 10 + 4\ )$

Ask how they can apply the Distributive Property for Multiplication with respect to Addition to arrive at a product.

$[\ 3 \times 14 = (\ 3 \times 10\ ) + (\ 3 \times 4\ )\ ]$

Emphasize this application of the Distributive Property using the array.



$(\ 3 \times 10\ ) + (\ 3 \times 4\ )$

Present the problem in vertical format

$$\begin{array}{r} 14 \\ \underline{\times\ 3} \end{array}$$

Children should apply the Distributive Property and find the product.

$$\begin{array}{rl} 14 & \\ \underline{\times\ 3} & \\ 12 & (3 \times 4) \\ \underline{30} & (3 \times 10) \\ 42 & (3 \times 14) \end{array} \quad \text{or} \quad \begin{array}{rl} 14 & \\ \underline{\times\ 3} & \\ 30 & (3 \times 10) \\ \underline{12} & (3 \times 4) \\ 42 & (3 \times 14) \end{array} \quad \text{or} \quad \begin{array}{rcl} 10 & + & 4 \\ \underline{\times} & & \underline{3} \\ 30 & + & 12 = 42 \end{array}$$

2. Present exercises involving no exchange, one exchange and the two exchanges.

$$\begin{array}{r} 32 \\ \underline{\times\ 3} \end{array} \qquad \begin{array}{r} 29 \\ \underline{\times\ 4} \end{array} \qquad \begin{array}{r} 71 \\ \underline{\times\ 6} \end{array} \qquad \begin{array}{r} 45 \\ \underline{\times\ 6} \end{array}$$

$$\begin{array}{r} 143 \\ \underline{\times\ 2} \end{array} \qquad \begin{array}{r} 128 \\ \underline{\times\ 4} \end{array} \qquad \begin{array}{r} 182 \\ \underline{\times\ 4} \end{array} \qquad \begin{array}{r} 149 \\ \underline{\times\ 4} \end{array}$$

3. Develop skill in multiplying larger numbers, one factor through 9; the other factor through 999.

Have children first estimate, then compute, then compare the product with the estimate.

For example, $4 \times 921 = n$

a. Estimation: $4 \times 900 = 3600$

b. Computation:

$$\begin{array}{r l} 921 & \\ \times\ 4 & \\ \hline 4 & (4\times1) \\ 80 & (4\times20) \\ 3600 & (4\times900) \\ \hline 3684 & (4\times921) \end{array}$$

Product may be read as:
36 hundred 84 or as:
3 thousand 6 hundred 84

c. Comparison:

3684 with 3600

4. Introduce the Concise Form for the algorithm.

$$\begin{array}{ccc} \text{A} & \text{B} & \text{C} \\ \begin{array}{r l} 312 & \\ \times\ 3 & \\ \hline 6 & (3\times2) \\ 30 & (3\times10) \\ 900 & (3\times300) \\ \hline 936 & (3\times312) \end{array} & \begin{array}{r} 312 \\ \times\ 3 \\ \hline 6 \\ 30 \\ 900 \\ \hline 936 \end{array} & \begin{array}{r} 312 \\ \times\ 3 \\ \hline 936 \\ \\ \\ \\ \\ \end{array} \end{array}$$

Present Forms A, B, then C.

Discuss differences in recording.

Compare products. Children note that Place Value of the digits in the short form is the same as in the long form.

Refer to form C. Children record product in expanded notation.
[900 + 30 + 6]

Compare Forms A, B, C again.
Note the economy in the use of the short form.

5. Extend to exercises involving Exchange.

Present forms A and B below simultaneously. If necessary demonstrate with arrays or with squared material first.

$$\begin{array}{rl@{\qquad\qquad}r} \text{A} & & \text{B} \\ 216 & & 216 \\ \times\ 3 & & \times\ 3 \\ \cline{1-1}\cline{3-3} 18 & (\ 3\times6\ ) & 648 \\ 30 & (\ 3\times10\ ) & \\ 600 & (\ 3\times200\ ) & \\ \cline{1-1} 648 & & \end{array}$$

Ask children:

Consider Form B. Why is "8" in ones place?

Why is 4 in tens place in the product in Form A?

What did we do to get 4 in tens place in Form B?

Where did we get the number added? [18 = 10 + 8]

Why is 6 in Form B in hundreds place?

6. Extend to exercises involving two and more exchanges.

Children may check by applying the Distributive Property.

$$\begin{array}{r@{\qquad\qquad\qquad}l} 937 & 4\ \times\ \ \ 7 = \ \ \ 28 \\ \times\ 4 & 4\ \times\ \ 30 = \ \ 120 \\ \cline{1-1} 3748 & 4\ \times 900 = 3600 \\ \cline{2-2} & 4\ \times 937 = 3748 \end{array}$$

7. Suggested practice exercises

$$\begin{array}{r@{\qquad\qquad}r@{\qquad\qquad}r@{\qquad\qquad}r@{\qquad}l} 419 & 635 & 302 & 892 & \\ \times\ 4 & \times\ 7 & \times\ 9 & \times\ 6 & \\ \cline{1-4} & 35 & & & \text{etc.} \\ & 210 & & & \\ & 4200 & & & \\ \cline{2-2} \end{array}$$

8. Apply to problem situations to provide more practice.

OPERATIONS

UNIT 24 - DIVISION OF WHOLE NUMBERS

$$\begin{tikzcd}
\text{Addition} \arrow[r] \arrow[d, leftrightarrow] & \text{Multiplication} \arrow[d, leftrightarrow] \\
\text{Subtraction} \arrow[r] & \text{Division}
\end{tikzcd}$$

NOTE TO TEACHER

In Multiplication two known
the problem is to find their product.

Factor x Factor = □

Another interpretation of multiplication with whole numbers involves rectangular arrays of elements,

x x x x x
x x x x x
x x x x x

where we know the number of elements in each row and the number of elements in each column
product is the number of elements in the array.

In Division two interpretations should be emphasized to parallel the two interpretations of multiplication:

1. When one factor and the product are known and the problem is to determine the missing factor
2. Where the total number of elements in the array is known and the number of rows (or columns) is known. The problem is to determine the number of columns (or rows).

In Division an array can not always be set up with the same number of elements in each row.

For example, 25 ÷ 4 set up in an array will have a remainder.

Given an array of 25 elements with 4 elements in a row the problem is to find the number of rows of 4.

This process may be shown as follows:

25	elements in the array
- 4	one row of 4 elements removed
21	elements remaining
- 4	another row of 4 elements removed
17	elements remaining
- 4	
13	elements remaining
- 4	
9	elements remaining
- 4	
5	elements remaining
- 4	
1	element remaining

By adding the number of times 4 elements have been removed we see that the quotient is 6 and the remainder is 1.

When children divide they remove from the dividend as many equivalent sets as they can.

Gradually they are encouraged to refine the process and remove a greater number of sets at one time.

Children express the remainder as a fractional number when a verbal problem indicates that it is reasonable to do so. It would not be reasonable to divide children into fractional parts.

They record a division with a remainder as:

$$\begin{array}{l} \quad\ 6 \text{ r } 1 \\ 4\overline{)25} \end{array}$$

Relate division to problem situations based on the children's experience. When situations are presented involving dollars and cents, children think in terms of the number of cents, $1.54 as 154 cents.

Suggested Problem Situations

Postage stamps purchased

Have $.72. Need four-cent stamps. How many stamps can be purchased? (Fours in 72)

P.T.A. Cake Sale

86 home baked cookies on hand. Packaged 5 in each waxed bag. How many bags filled with cookies? (Fives in 86)

Cup cakes sold at 7 cents each. $1.54 collected. How many cup cakes sold? (Sevens in 154)

Picture Postcards in album

Trip to Washington D.C. - 138 cards bought. 6 on each page in the album. How many pages completely filled? (Sixes in 138)

Dance Formations

184 children in gym. Sets of 8 children formed. How many sets formed? (Eights in 184)

Suggested Provision for Differences in Ability

Less Mature	More Mature
Deal with divisors of 3, 4, 5 and perhaps 6 at first.	Deal with all divisors including 6, 7, 8, 9.

Record as few or as many computations as they need, by using groups smaller than 10, or groups of 10.	Are encouraged to shorten their computation by using multiples of 10.
Later deal with divisors of 6, 7, 8, 9 - quotients in the tens, twenties and thirties, e.g., $8\overline{)97}$ $6\overline{)198}$ $7\overline{)158}$	Deal with more difficult division situations - quotients in the fifties, sixties, seventies. $4\overline{)289}$ $6\overline{)321}$ $3\overline{)204}$ $7\overline{)448}$
Are encouraged to shorten the computation by using multiples of 10.	

Objectives: To help children:

Use algorithms to find quotients
Refine the techniques of the Operation of Division.

TEACHING SUGGESTIONS

Evaluation

Before division is introduced, the teacher finds out whether the children:

Understand the meaning of the numbers to be used.
Understand the meaning of subtraction and multiplication.
Have acquired skill in the operations of multiplication and subtraction.
Have achieved automatic response to the multiplication, subtraction and division facts they will use.
Have the ability to multiply numbers through 9 by 10 and multiples of 10.

Dividends through 999: Divisors through 9

1. No Remainder

Present a problem situation such as:

192 children are arranged into sets of 8 for a dance.
How many sets will there be?

Record as a mathematical sentence. $n \times 8 = 192$

Discuss the action involved (separating 192 into subsets of 8).

Record the symbols on the chalkboard. $8\overline{)192}$

(Read as "How many eights in 192?")

Discuss with the children the solution of the problem by removing successive sets of eight.

$$\begin{array}{r} 192 \\ \underline{-\ 8} \\ 184 \\ \underline{-\ 8} \\ 176 \\ \underline{-\ 8} \\ \text{etc.} \end{array}$$

Encourage children to see that it would be more efficient to remove several sets of 8 at a time. Each child may choose the number of sets of 8 he will take out.

Select one of the suggestions, e.g., 5 eights, and use it to develop the division algorism.

After the product (40) has been recorded, question children about its meaning.

Elicit from them that there are still 152 children to be divided into sets of 8.

Children continue to remove sets of 8 until zero is the remainder.

Total the number of eights and record the quotient.

$$\begin{array}{rrl} & 8\overline{)192} & \\ \text{5 eights} = & \underline{40} & \\ & 152 & \text{to be divided} \\ \text{6 eights} = & \underline{48} & \\ & 104 & \text{to be divided} \\ \text{10 eights} = & \underline{80} & \\ & 24 & \text{to be divided} \\ \underline{3}\ \text{eights} = & \underline{24} & \\ \text{24 eights} & 0 & \end{array}$$

Solution: 24 sets of children.

Solve the same problem beginning with other sets of eights. The illustrations below show some possible variations.

$$\begin{array}{rr} \text{? eights} = & 192 \\ & 8\overline{)192} \\ \text{8 eights} = & \underline{64} \\ & 128 \\ \text{8 eights} = & \underline{64} \\ & 64 \\ \underline{8}\ \text{eights} = & \underline{64} \\ \text{24 eights} & 0 \end{array}$$

Solution-24 eights

$$\begin{array}{rr} \square \times 8 = & 192 \\ & 8\overline{)192} \\ 5 \times 8 = & \underline{40} \\ & 152 \\ 10 \times 8 = & \underline{80} \\ & 72 \\ \underline{9} \times 8 = & \underline{72} \\ \text{24 eights} & 0 \end{array}$$

Solution-24 eights

$$\begin{array}{rr} \square\ 8 = & 192 \\ & 8\overline{)192} \\ 10 \times 8 = & \underline{80} \\ & 112 \\ 10 \times 8 = & \underline{80} \\ & 32 \\ \underline{4} \times 8 = & \underline{32} \\ \text{24 eights} & 0 \end{array}$$

Solution-24eights

Solve other problems with children in a variety of ways. Then present problems for each child to solve in his own way.

2. Remainder

Present a problem situation:

We are making booklets for Social Studies. Seven pieces of colored paper are needed for each booklet. How many booklets can we make if we have 200 sheets of paper?

Follow the same procedure suggested for division without remainders.

$\square \times 7 = 200$

$$\begin{array}{rcrcr}
 & & & & 7\overline{)200} \\
7 & \times & 7 & = & 49 \\
 & & & & \overline{151} \\
10 & \times & 7 & = & 70 \\
 & & & & \overline{81} \\
10 & \times & 7 & = & 70 \\
 & & & & \overline{11} \\
\underline{1} & \times & 7 & = & \underline{7} \\
28 \text{ sevens} & & & & 4
\end{array}$$

Solution: 28 booklets and 4 extra sheets

Through questioning, lead the children to conclude:
That another set of seven cannot be formed;
That the 4 represents four sheets of paper left from the 200 (the dividend).

Present other divisions without and with remainders.

Develop a More Concise Form

1. Present a problem situation such as: 139 books collected for the Veteran's Hospital, to be tied in bundles of 6 books each. How many bundles will there be?

   As each partial quotient is recorded in A, B, and C compare all forms, asking such questions as:

   What does the 7 represent? What does the 8 represent?
   What is the 97? What is the 49? etc.

A

$$\begin{array}{rcr} & & 6\overline{)139} \\ 7 \times 6 & = & \underline{42} \\ & & 97 \\ 8 \times 6 & = & \underline{48} \\ & & 49 \\ \underline{8 \times 6} & = & \underline{48} \\ 23 \text{ sixes} & & 1 \end{array}$$

B

$$\begin{array}{rcl} 6\overline{)139} & & \\ \underline{42} & = & 7 \times 6 \\ 97 & & \\ \underline{48} & = & 8 \times 6 \\ 49 & & \\ \underline{48} & = & \underline{8 \times 6} \\ 1 & & 23 \text{ sixes} \end{array}$$

C

$$\begin{array}{r|l} 6\overline{)139} & \\ \underline{42} & 7 \\ 97 & \\ \underline{48} & 8 \\ 49 & \\ \underline{48} & \underline{8} \\ 1 & 23 \end{array}$$

Solution: 23 Remainder 1

2. Present other division situations using all forms.
Then encourage the children to use the concise form only.

3. Use 10 or multiples of 10 to arrive at quotients.

Before using 10 or multiples of 10 to find a quotient, reinforce multiplying by whole decade numbers.

10 sixes, 20 sixes, 30 sixes, etc.

Use a familiar problem situation: 139 books collected for the Veteran's Hospital, to be tied in bundles of 6 books each. How many bundles will there be?

At first children find the solution by removing successive groups of 10 sixes.

$$\begin{array}{rr|l} & 23 & \\ & 6\overline{)139} & \\ & \underline{60} & 10 \\ \text{Still to be divided} \rightarrow & 79 & \\ & \underline{60} & 10 \\ \text{Still to be divided} \rightarrow & 19 & \\ & \underline{18} & \underline{3} \\ & 1 & 23 \end{array}$$

Solution: 23 sets and 1 extra book

When children develop facility in multiplying with larger multiples of 10 (20, 30, 40, etc.), they shorten the computation by removing groups of 20 sixes, 30 sixes, 40 sixes, etc.

For the problem suggested above question children and record the proper responses.

Will there be as many as 10 sixes? [Yes; 10 sixes = 60]

Will there be as many as 20 sixes? [Yes; 20 sixes = 120]
Will there be as many as 30 sixes? [No; 30 sixes = 180]

Then how many sets of sixes shall we remove first? [20 sixes]

How many books will that be? [120]

Have we used up all the books?

How many are left to be divided?

Continue until no more sets of six can be removed.

Discuss remainder.

$$\begin{array}{r|r} \multicolumn{1}{r}{23} & \\ 6\overline{)139} & \\ \underline{120} & 20 \\ 19 & \\ \underline{18} & \underline{3} \\ \hline 1 & 23 \end{array}$$

Solution: 23 sets and 1 extra book.

Present many other divisions. Encourage those children who are able to use the shortened computation (multiples of ten).

$$\begin{array}{r|r} \multicolumn{1}{r}{35} & \\ 8\overline{)284} & \\ \underline{160} & 20 \\ 124 & \\ \underline{80} & 10 \\ 44 & \\ \underline{40} & \underline{5} \\ \hline 4 & 35 \end{array} \quad \text{or} \quad \begin{array}{r|r} \multicolumn{1}{r}{35} & \\ 8\overline{)284} & \\ \underline{240} & 30 \\ 44 & \\ \underline{40} & \underline{5} \\ \hline 4 & 35 \end{array}$$

Solution: 35 R 4

$$\begin{array}{r|r} \multicolumn{1}{r}{63} & \\ 6\overline{)381} & \\ \underline{120} & 20 \\ 261 & \\ \underline{120} & 20 \\ 141 & \\ \underline{120} & 20 \\ 21 & \\ \underline{18} & \underline{3} \\ \hline 3 & 63 \end{array} \quad \text{or} \quad \begin{array}{r|r} \multicolumn{1}{r}{63} & \\ 6\overline{)381} & \\ \underline{120} & 20 \\ 261 & \\ \underline{240} & 40 \\ 21 & \\ \underline{18} & \underline{3} \\ \hline 3 & 63 \end{array} \quad \text{or} \quad \begin{array}{r|r} \multicolumn{1}{r}{63} & \\ 6\overline{)381} & \\ \underline{300} & 50 \\ 81 & \\ \underline{60} & 10 \\ 21 & \\ \underline{18} & \underline{3} \\ \hline 3 & 63 \end{array}$$

Children will need continuous reference to experience situations.

Many texts contain problems based on children's experience. These should be carefully selected.

Encourage children to shorten the recording by using larger and fewer multiples of ten.

```
           75                          75                       75
From:    6)452         Encourage     6)452         Advance    6)452
           120 | 20                    300 | 50       to        420 | 70
           332 |                       152 |                     32 |
           240 | 40                    120 | 20                  30 |  5
            92 |                        32 |                    -----------
            60 | 10                     30 |  5                   2 | 75
            32 |                       -----------
            30 |  5                      2 | 75
           -----------
             2 | 75
```

Solution: 75 R 2

Dollars and Cents

Present a problem situation such as: How many air mail stamps at \$.08 each can be bought for \$1.75?

Record and discuss the following forms:

$0.08)$\overline{\$1.75}$	8 cents)$\overline{175}$ cents	8)$\overline{175}$
How many \$.08 in \$1.75?	8 cents in 175 cents?	eights in 175?

Compute as Shown

```
    21
  8)175
    160 | 20
     15 |
      8 |  1
   ----------
      7 | 21
```

Solution: 21 stamps can be purchased.
7 cents left over.

Discuss the meaning of the remainder, 7 and the quotient 21.

Terminology

Discuss the meaning of dividend, divisor, quotient and remainder. Relate each term to the numbers of the specific problem.

139 books are to be tied in bundles of 6. How many bundles will there be?

```
    23
  6)139
    120 | 20
     19 |
     18 |  3
   ----------
      1 | 23
```

What does the 139 represent? [The number of books to be divided]

What does the 6 represent? [The number of books in each bundle; the number of each set.]

What does the 23 represent? [The number of equivalent sets; the number of bundles.]

What does the 1 represent? [An extra book; there are not enough books to make another set of 6]

Continue the above type of questioning. Use the terms for division. For example:

Which is the divisor? [6]

What does the 6 tell us? or What is the 6 called? [the divisor]

What does the divisor tell?

GEOMETRY AND MEASUREMENT

UNIT 25 - GEOMETRY: PLANES; SIMPLE CLOSED CURVES

NOTE TO TEACHER

Understanding of the meaning of a plane precedes the understanding of properties of polygons and circles and other plane figures.

A point is an idea just as a number is. It can be represented by a dot, just as a number is represented by a numeral. It has no dimensions.

A line is an idea. Its' representation; has only one dimension, length.

Figure 1

A plane is an idea. Its' representation:

has only 2 dimensions, length and width.

Figure 2

Planes may be thought of as special sets of points in space, which is considered to be the set of all points.

A plane, being a set of points, is a subset of space. (see Figure 2)

A plane can be thought of as extending without limit in space.

The following diagram indicates this.

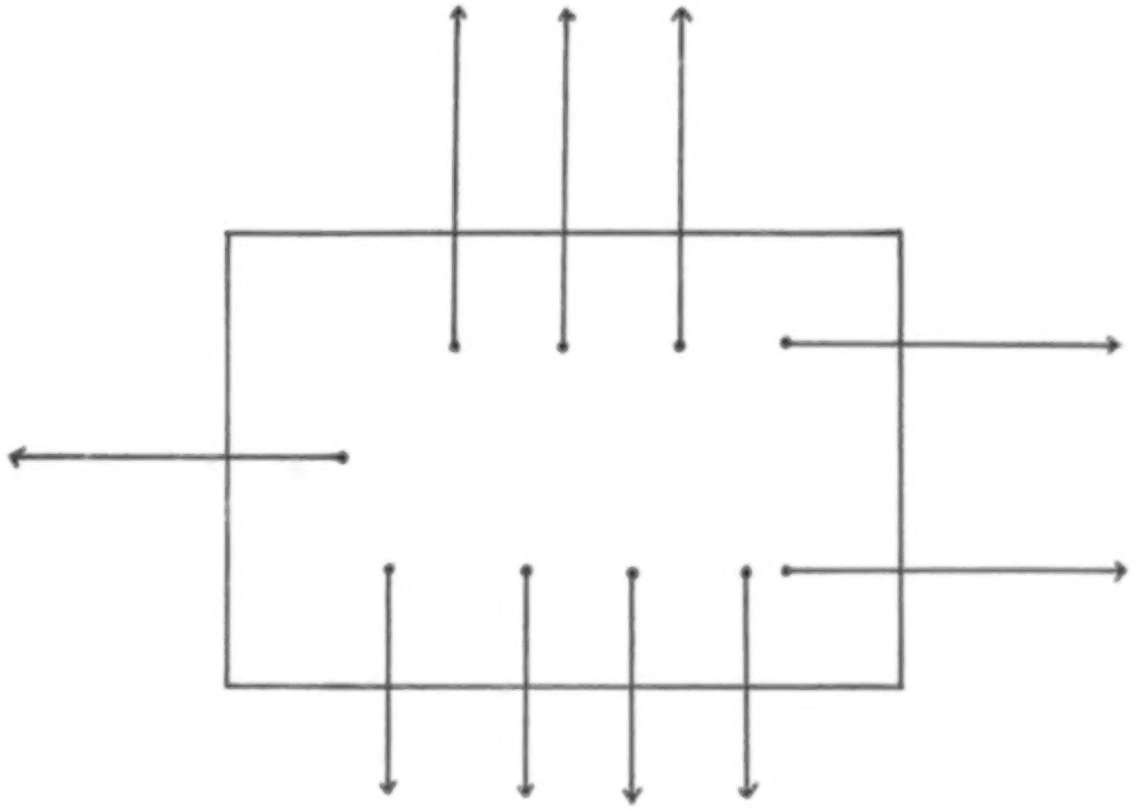
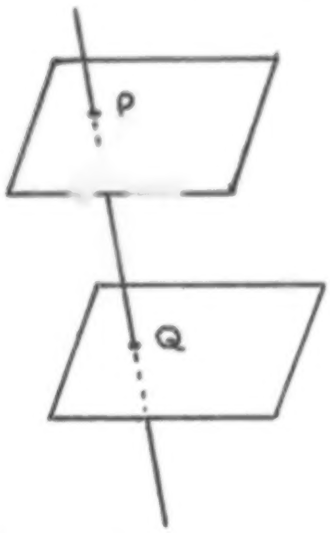
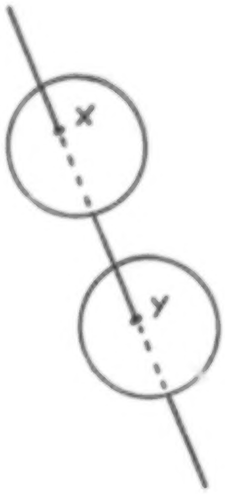


Figure 3 (A vertical plane)

Any plane contains an infinite number of lines. Therefore, it contains an infinite number of points. The surface of a table may be thought of as a representation of a segment of a plane.

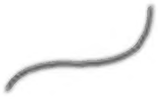
Each diagram below represents segments of 2 planes intersected by a line.

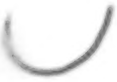


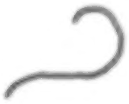


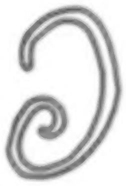
Lines A, B, C, D below are curves in one plane. (plane curves)

A simple plane curve may be represented as a set of points which can be drawn without lifting pencil from paper and without intersecting itself.

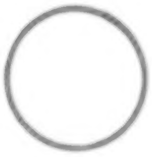

A

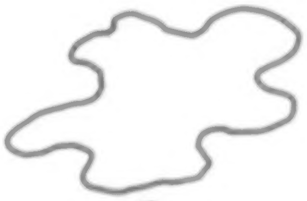

B


C


D

A simple closed plane curve is a curve in a plane whose starting and ending points are the same and which does not intersect itself at any point. For example:


E


F

D, E and F are simple closed curves.

Objective: To develop concepts of planes; simple closed curves.

TEACHING SUGGESTIONS

Planes

1. Reinforce concepts of:

A point

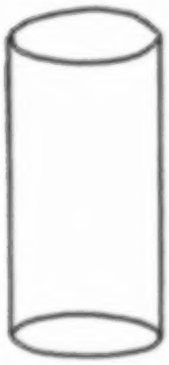
Sets of points

2. Tell children to imagine a pegboard extending without end in all directions. Imagine even more holes then there are in the pegboard. Think of the holes as representing points. All of the points form a set. This set of points on a flat surface is called a plane.

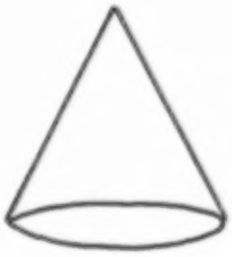
   A plane is therefore a set of points extending indefinitely in all directions.

   Imagine a crushed piece of paper; the ocean; a bumpy road. These are not flat surfaces. These are not in one plane. But these are also surfaces and sets of points.

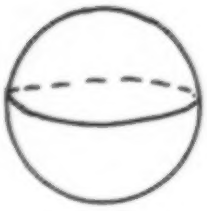
3. A plane is a special kind of surface. It has a special property that other kinds of surfaces, curved surfaces, such as conical surfaces, cylindrical surfaces, spherical surfaces do not have



Cylindrical Surface



Conical Surface

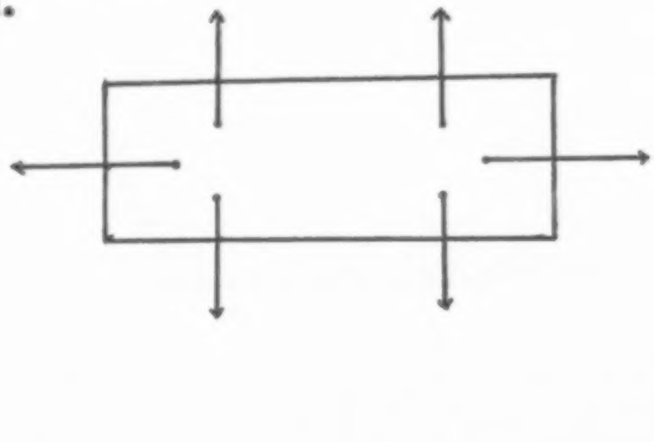


Spherical Surface

   namely: The straight line joining any two points in a plane lies wholly in the plane.

4. Summarize:

   A plane is a set of points.
   A plane is a surface. Other kinds of surfaces are indicated above.
   A plane has a special property. (See underlined statement above.)
   A plane like a set of points can only be imagined.
   A plane has no thickness. It extends indefinitely in all directions except depth.



A plane is a special set of points in space.

Provide each child with a mirror and an index card. Ask children to adjust their mirror so that the card is reflected in the mirror.

They should see that the reflection of the card has negligible thickness, only length and width.

5. Ask children to name different plane surfaces in the classroom.

[desk tops, floor, ceiling, etc.]

6. Tell children that the surfaces mentioned represent only segments of planes. Compare with line segment.

Ask children:

How many lines can be drawn in a plane?
How many points are there in a plane?
How far can a plane be extended?
To describe in their own words the difference between space and a plane; a segment of a plane and a plane.

[A plane is a subset of space; a segment of a plane is a subset of a plane.]

To name some subsets of a plane.

[points, lines, simple closed curves, etc.]

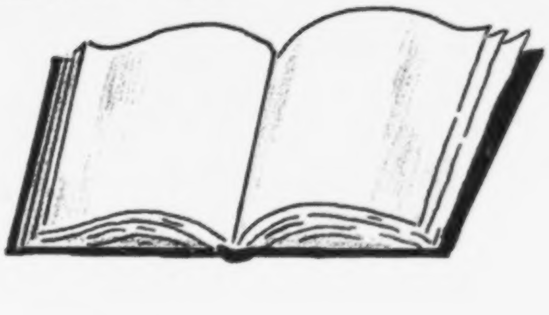
7. Draw a line and represent a point outside the line by a dot. How many planes can contain this line and the point?

[one]

Indicate three points that are not in a straight line. How many planes can contain these 3 points?

[There is only one plane that contains three non-collinear points]

How many planes can be determined by one line?
Explain or illustrate this.



use leaves of a book

What is the intersection of two different planes?

[a line]

8. Develop with children the "linearity" or "flatness" property of a plane surface. Bring into class cylindrical, conical and spherical solids and try to lay a ruler flat on the surface. It can be done:

   a. Sometimes with a cylinder or a cone
   b. Never with a sphere
   c. Always with a plane surface.

   Talk about how a plasterer creates a flat wall by moving a long rectangular instrument in all directions to create the plane surface.

   Do not try to verbalize this definition or property of flatness.

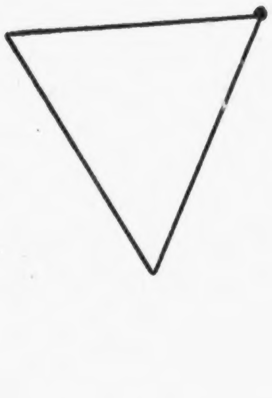
Simple Closed Plane Curves

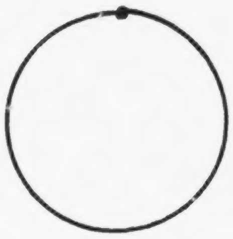
1. Reinforce concept of curve.

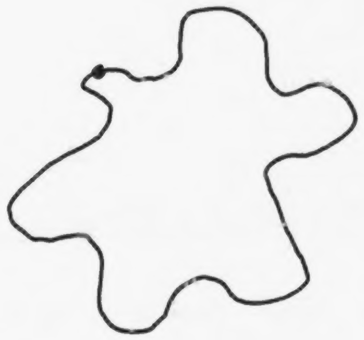
2. Direct children to draw a curve such that:

   The pencil must not be lifted from the paper.
   The starting point and point at which they end must be the same.
   The line must not cross itself at any point.

   For example:

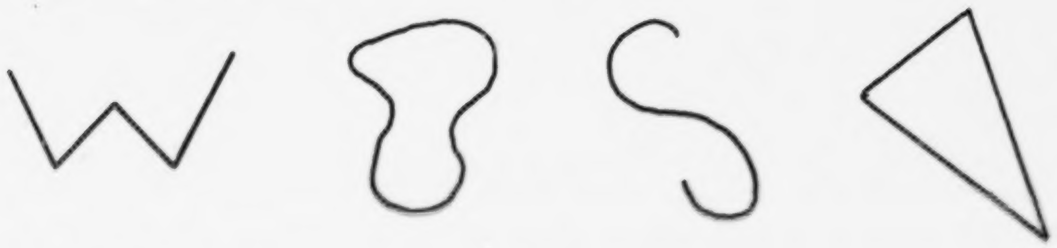






3. Children note that the figures enclose parts of a plane and separate it into an inside and an outside.

4. Draw figures such as the following:



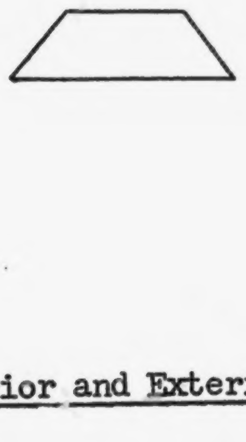
Children discuss differences among curves.

Tell children:

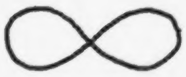
> A closed figure that can be drawn on a plane without lifting the pencil from the paper and that does not intersect itself is called a simple closed figure.

5. Children identify the simple closed curves in the diagrams below.

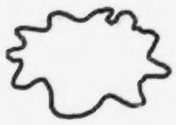
A.



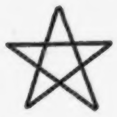
B.



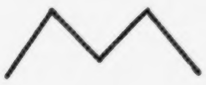
C.



D.

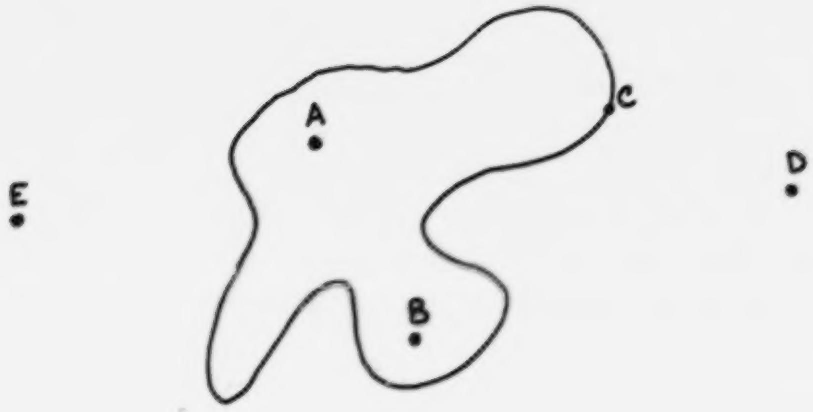


E.



Interior and Exterior of Curves

1. Have children draw a simple closed curve, then place dots representing points anywhere on their papers, and label the dots A, B, C, D, etc.

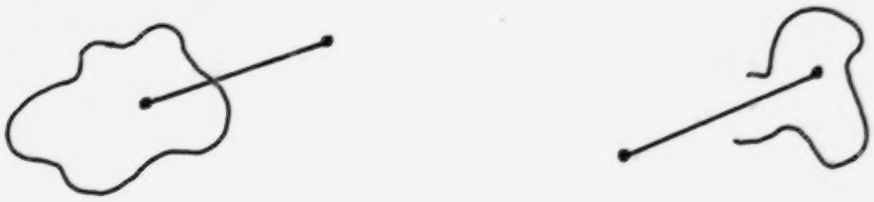


Ask the children to describe the location of point A, point B, point C, etc.

[inside the curve, on the curve, outside the curve]

Introduce the words: interior, exterior, boundary

2. Draw another simple closed curve on the chalkboard. Have children place a dot on the interior of the curve, another on the exterior, then connect the two dots with a line. They do the same with an open curve.



Ask children:

What separates the interior of the simple closed curve from the exterior?

[the boundary]

Into how many regions does the boundary separate the plane?

[Two regions: the interior of the curve,
the exterior of the curve]

Have children note that the boundary of the two regions is the closed curve.

They note that an open curve that has two endpoints does not separate a plane into two regions.

GEOMETRY AND MEASUREMENT

UNIT 26 - GEOMETRY: RAYS; ANGLES

NOTE TO TEACHER

In **Unit** 4 we defined space as the set of all points.

Physical space considered as the set of all points, means the set of all possible locations in the universe. A moving airplane is constantly occupying different sets of points in space.

We have previously discussed some sets of points as locations in space, as curves, as lines, as line segments, as planes, as other kinds of surfaces.

In this Unit we extend concepts of lines.

Rays

A point "A" of a line separates the line into two parts.

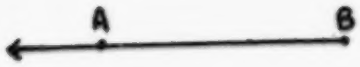
A ray is a part of a line that is composed of a point, such as A, and all points on that line on only one side of A.

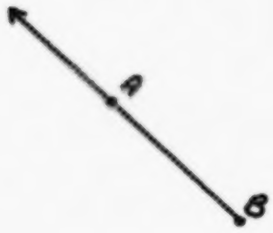
This is a drawing of a ray with endpoint A

The symbol for ray is $\longrightarrow$. For example,

is symbolized as $\overrightarrow{AB}$

Should the endpoint be B with the ray extending in another direction thus, for



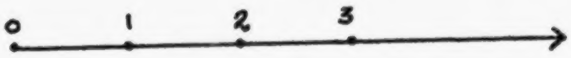


the symbolization would be $\overrightarrow{BA}$.

$\overrightarrow{BA}$ indicates that B is the endpoint.
Every ray has just one endpoint.

Number Ray

Part of a number line is an example of a number ray.



Angle

An angle consists of two rays with a common endpoint called its vertex. The rays are called the sides of the angle.

Objectives: To extend concepts of lines to include rays.
To develop concepts of angles.

TEACHING SUGGESTIONS

Ray

1. Discuss "Living in the Space Age" with children.
   Use newspaper articles, books, magazines and children's information to talk about Satellites, Astronauts, Landings on the Moon.

   Ask children:

   What information do we need for travel and communication in space?

   [Answers will vary. Some may be: direction, speed, location]

   How can we learn about these things we cannot see?

   [Use imagination, known facts, representations of things, models]

   What do you think space is? [Answers will vary]

2. Tell children that space is considered to be the set of all points everywhere.

   Review concepts of space developed in Unit 4.

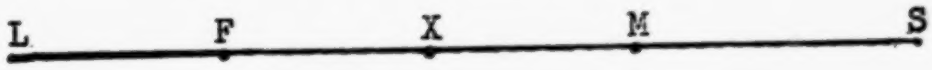
3. Review concept of line and line segment as special sets of points.

4. Direct children to:

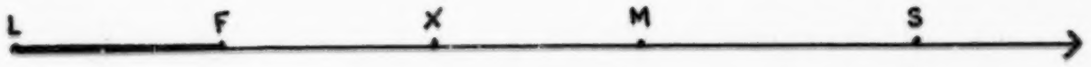
   Draw a line segment. Teacher draws on board.

Label endpoints L, F Name the segment $\overline{LF}$.

Keep L as the endpoint and draw successively longer line segments and label each.



5. Teacher draws a ray with the same endpoint, L, but extending indefinitely.

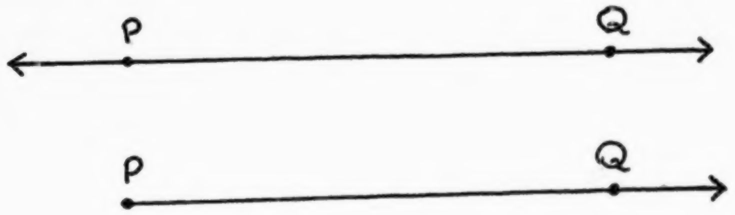


Ask children:

What is the difference between the two drawings?

If we symbolize the first drawing as $\overline{LS}$ can you suggest a way of symbolizing the second drawing?

Teacher may draw the following two figures on the chalkboard.



Ask children:

What do we call the upper drawing?

[line PQ]

How do we symbolize line PQ? [$\overleftrightarrow{PQ}$]

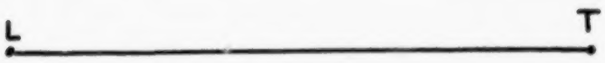
How does the lower drawing differ from $\overleftrightarrow{PQ}$?

Tell children that the lower diagram is called a ray. Discuss the term "ray" in life situations (light ray, x-ray, etc.).

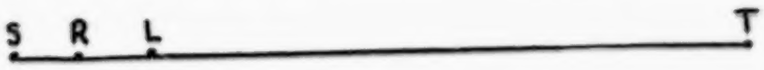
Ask children: To describe this ray [This ray has an endpoint. It extends indefinitely in one direction only.]

To explain why a ray is a subset of a line.

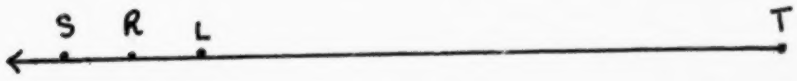
6. Children draw another line segment with endpoints L and T.



They use T as the endpoint and draw successively longer line segments in the opposite direction.



Ask children to extend the line segment $\overline{TS}$ so that a ray results.



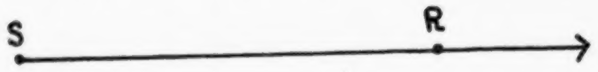
Children note that T is the endpoint and the ray extends indefinitely to the left of T.

7. Help children to verbalize the meaning of a ray.

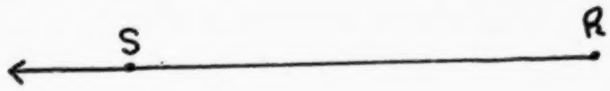
> [A ray has only one endpoint and it extends indefinitely in one direction.]

8. Tell children that the symbol for ray is $\rightarrow$. The direction of the ray is indicated by the two letters used.

For example, $\overrightarrow{SR}$ indicates that the ray extends to the right of the endpoint, S



$\overrightarrow{RS}$ indicates that the ray extends to the left of endpoint R



$\overrightarrow{SR}$ and $\overrightarrow{RS}$ are different rays having different endpoints. The name of the endpoint is written first. Every ray has only one endpoint.

SUGGESTED PRACTICE EXERCISES

1. Consider this drawing of a ray

   What is the endpoint of $\overrightarrow{DE}$? [D]

   Is $\overrightarrow{DE}$ the same as $\overrightarrow{ED}$? Explain

2. Draw a diagram of a ray. Name it using the symbol for ray.

3. Mark a point on your paper. Label it.

   Draw 5 rays with that endpoint.
   How many rays can be drawn with that endpoint?

4. Draw three different pictures that occupy space.

5. Write what you think space is. Explain

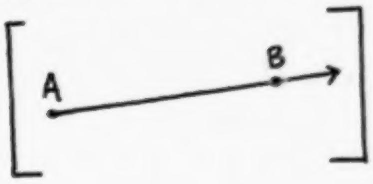
Angles

Direct children:

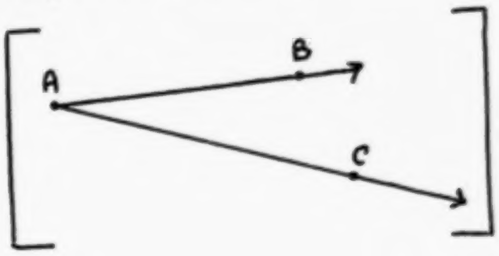
Mark a point on your paper. [•]

Mark it A. [$\overset{A}{\bullet}$]

Use A as an endpoint and draw a ray.



From the same endpoint A draw another ray.



Consider these figures

Note that there are two rays.
Note that the rays have a common endpoint.
Name the two rays.

Tell children that:

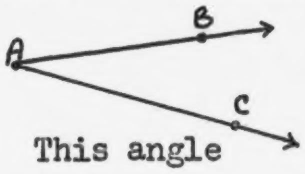
Two rays with a common endpoint form an angle.

Each ray is called a side of the angle.

The common endpoint is called the vertex of the angle.

The symbol for angle is " ∠ ".

Angles may be named in several ways. The letter naming the endpoint is placed in the middle.



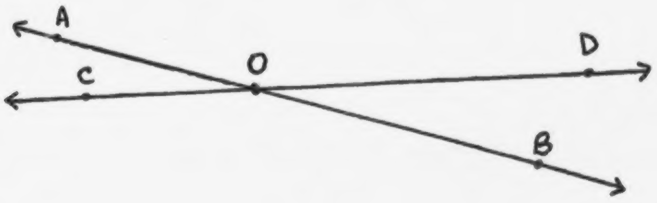
This angle may be named ∠BAC or ∠CAB.

Talk about an angle as the union of two rays.

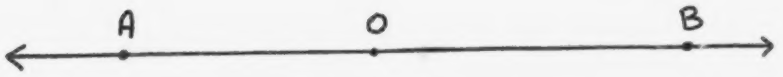
Direct children to:

Move the hands of a clock to indicate angles.
Tell where is the vertex of these angles.
Describe other objects that represent angles.

Name 4 angles that are formed when 2 lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at point O.

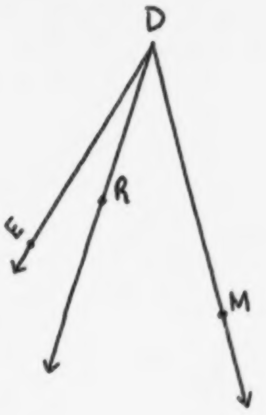


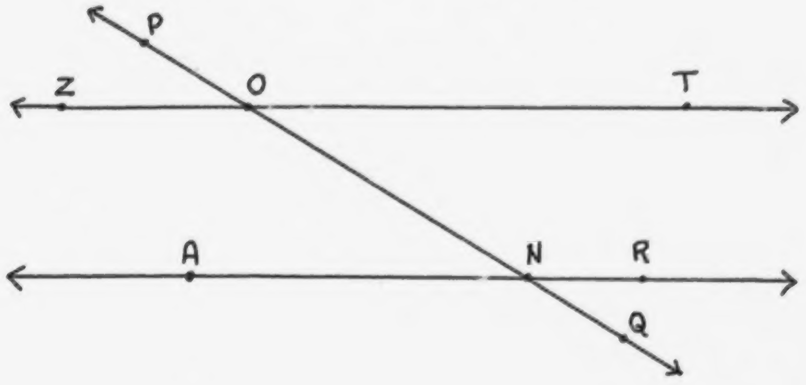
Name the angle formed by the 2 opposite rays meeting at vertex O in the diagram below.



(Here $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are parts of line $\overleftrightarrow{AB}$)

Name all the angles shown in the figures below. Be careful! Find as many angles as you can in each figure.





Direct children to point out the intersection of lines in the classroom that form different angles.

[angles formed by the binding of an open book.]

Tell children that angles shaped like a square corner are called right angles.

Ask children to draw right angles, angles less than right angles (acute), angles greater than right angles (obtuse).

GEOMETRY AND MEASUREMENT

UNIT 27 - GEOMETRY: POLYGONS; *EXPERIMENTAL GEOMETRY

NOTE TO TEACHER

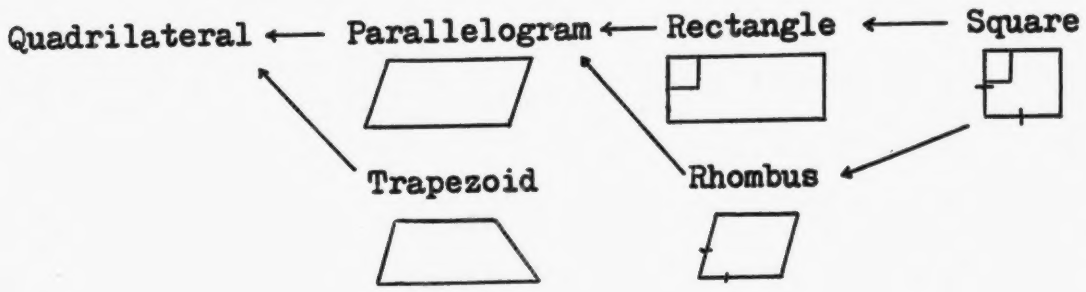
From a study of planes, line segments and simple closed curves we turn next to polygons.

A polygon is a simple closed curve which is a union of line segments. Polygon is the general name given to a family of geometric figures.

A triangle is a polygon consisting of the union of three line segments. We say it has 3 sides.

A quadrilateral is a polygon consisting of the union of four line segments; a pentagon, the union of five line segments; etc.

Each of the members of the set of polygons may again be subdivided by considering the kinds of angles and line segments it contains. Thus among the subset of polygons called quadrilaterals, there are parallelograms and trapezoids; some special parallelograms are rectangles and squares.



Objectives: To help children develop concept of polygons as simple closed curves.

To help children identify some types of quadrilaterals by their characteristics.

TEACHING SUGGESTIONS

1. Polygons

Ask children to:

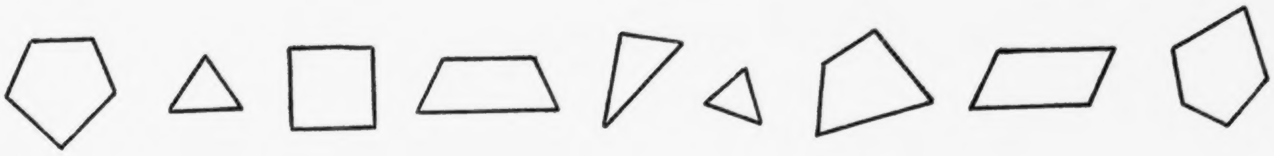
Draw a simple closed curve
Draw simple closed curves which are the union of 3 line segments, 4 line segments, 5 line segments, etc.

Tell children that:

Simple closed figures consisting of line segments are called polygons.
Polygons composed of 3 line segments are called Triangles.
Polygons composed of 4 line segments are called Quadrilaterals.
The line segments are called the sides of the polygon.

Ask children:

What is the name for a polygon containing 3 sides? [Triangle]
4 sides? [Quadrilateral] 5 sides? [Pentagon] etc.
To identify the following polygons and explain



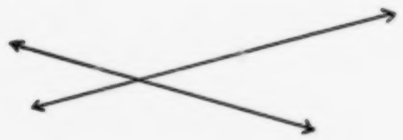
What is the greatest number of sides a polygon can have?
What is the least number of sides a polygon can have? Explain[3]

2. Special Kinds of Quadrilaterals

Parallelograms

Tell children that:

> Parallel Lines are lines in the same plane that do not intersect. They are everywhere equidistant.
>
> Non-parallel Lines in a plane are lines that do intersect. How many points of intersection do 2 non-parallel lines have? [One]



Discuss meaning of parallelogram.

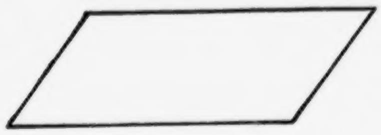
> A Parallelogram is a quadrilateral whose opposite sides are parallel. The opposite sides also have the same length.

Review meaning of rectangle:

> [Rectangles have 4"square corners,"(right angles) and the opposite sides are parallel. A rectangle is a special kind of parallelogram.]

Ask children:

> Is every rectangle a parallelogram? [Yes] Explain.
>
> Is every parallelogram a rectangle? [No] Explain.



Reinforce understanding of:

Right Angle:	A right angle is an angle shaped like a square corner. Name some objects in the room that have right angles.
Square:	A square is a quadrilateral having 4 sides of equal length and 4 right angles.

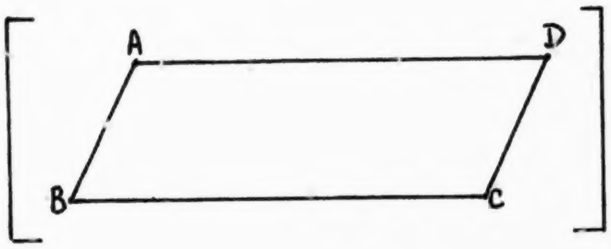
Discuss:

> A square as a special type of rectangle, and a special kind of parallelogram.
> Compare a square with other rectangles.
> When is a rectangle not a square?

EVALUATION AND / OR PRACTICE

SUGGESTED EXERCISES

1. Locate on your paper points A, B, C, D like these below.
   Draw $\overline{AB}$, $\overline{AD}$, $\overline{CB}$ and $\overline{DC}$

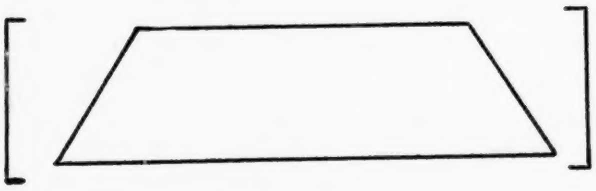


2. Which of these are names for this figure? [a, b, d]

   a. simple closed curve
   b. polygon
   c. triangle
   d. quadrilateral

3. Draw a picture of a quadrilateral without any parallel sides.

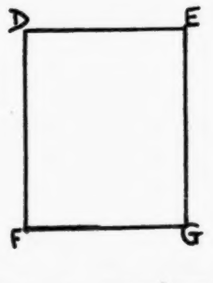
   Draw a picture of a quadrilateral with only two parallel sides.



   We call this a Trapezoid.

4. Name two special types of quadrilaterals. [parallelogram, rectangle]

5. What are the properties of parallelogram DEFG?



[$\overline{DE}$ and $\overline{FG}$ are parallel; $\overline{DF}$ and $\overline{EG}$ are parallel; all angles are right angles; it is a rectangle; etc.]

6. Why is a square a rectangle?

7. Is every square a parallelogram? If this is true, what must be true of the sides of a square?

8. Have children explore ways of drawing a rectangle, a square, a parallelogram.

9. Draw a parallelogram ABCD (as in exercise 1). Measure $\overline{AB}$ and $\overline{DC}$. What seems to be true? Measure $\overline{AD}$ and $\overline{BC}$. What seems to be true?

10. Draw a quadrilateral which is not a parallelogram. Measure pairs of opposite sides. Try to draw one with:

    a. only one pair of opposite sides equal in length
    b. with both pairs equal in length.

["a" can be done but for "b" it must be a parallelogram.]

11. Additional exercises may be found in textbooks.

*EXPERIMENTAL GEOMETRY (Optional)

NOTE TO TEACHER

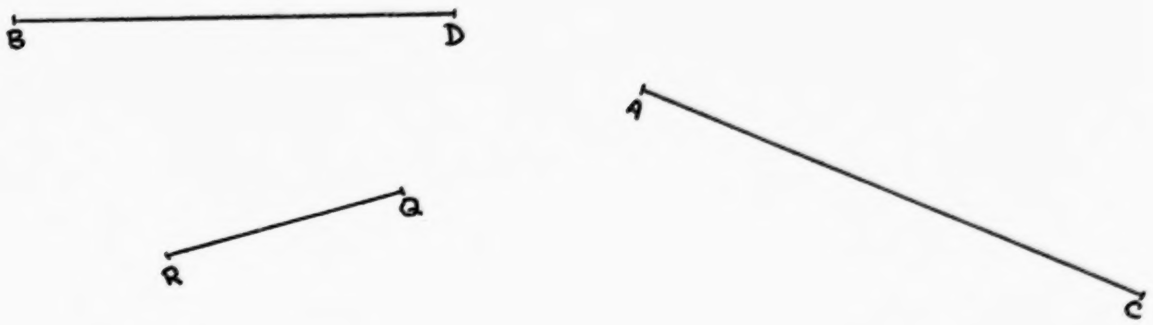
The Geometry in this section which we will call "Experimental Geometry" will be limited to further exploration of polygons.

Its purpose is twofold:
- To extend children's reasoning power
- To extend the use of measurement in Geometry so that children may discover some additional properties of geometric figures.

TEACHING SUGGESTIONS

Experimenting With Triangles

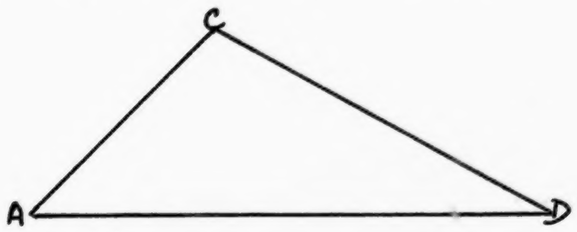
1. Have children draw line segments of varying lengths. For example:



Children should use rulers to find the mid-point of each line. (Practice in dividing a fractional number by 2 may have to be included for some children.)

2. Have children:

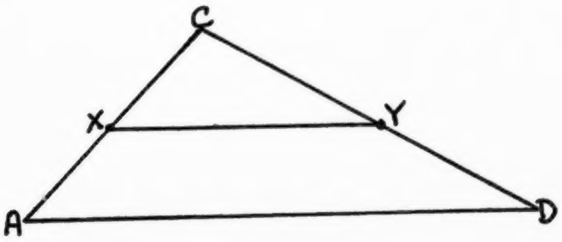
Draw any triangle as below, and label the vertices.



Use a ruler to find the mid-point of $\overline{AC}$. Label that point X.

Use a ruler to find the mid-point of $\overline{CD}$. Label that point Y.

Draw $\overline{XY}$



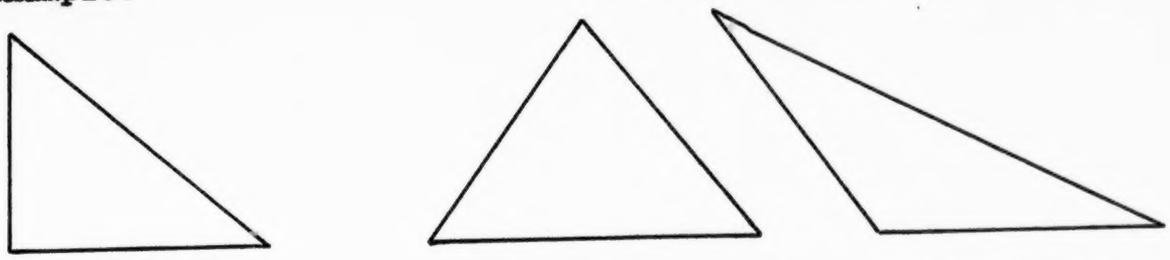
Measure $\overline{XY}$

Measure $\overline{AD}$

Ask children: What seems to be true about the relationship between $\overline{XY}$ and $\overline{AD}$?

[The measure of the length of $\overline{XY}$ is $\frac{1}{2}$ the measure of the length of $\overline{AD}$.]

3. Children should follow the same procedure using different triangles. For example:



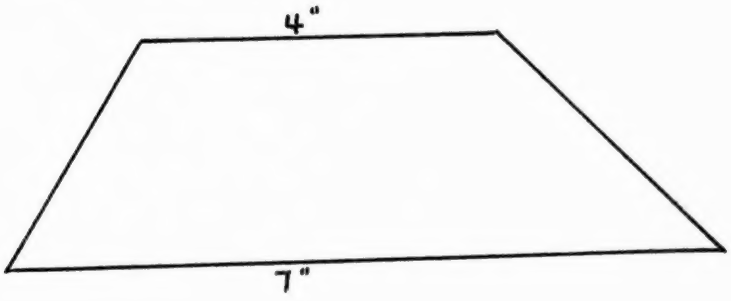
Right Triangle Equilateral Triangle Obtuse, etc.

They find out whether the same relationship holds true.

Median of a Trapezoid

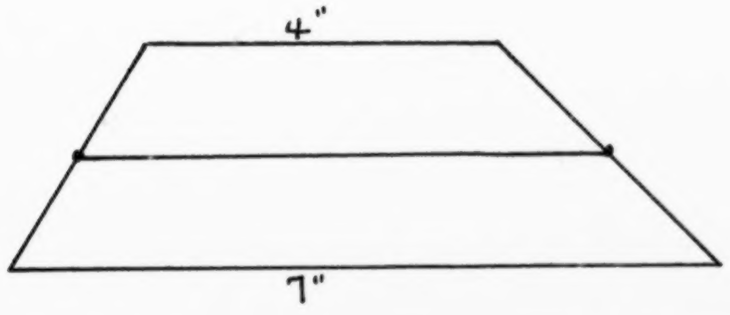
1. Reinforce meaning of a trapezoid.

2. Have each child draw a trapezoid whose parallel sides are whole numbers, and mark the length of those sides. For example:



3. Have the children find the mid-point of each of the two non-parallel sides.

They connect the points. We will call this



line the center line of the trapezoid.
Children measure the length of this center line.

4. On the chalkboard, make a chart of the various measurements that the children have obtained.

	Upper Base	Lower Base	Center Line
John's Trapezoid	1"	2"	1½"
Bob's Trapezoid	2"	3"	2½"
Mary's Trapezoid	2"	4"	3"
Ann's Trapezoid	1"	3"	2"

etc.

Question: Can you discover any relationship between the length of the center line and the lengths of the other two sides?

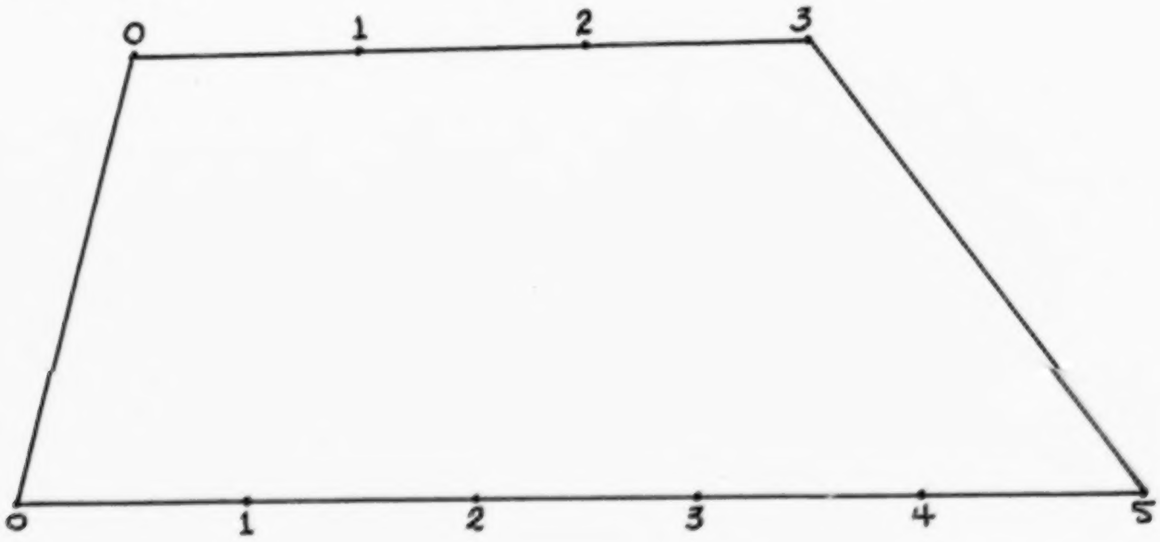
If we add the lengths of the other two sides, what is the relationship of this sum to the length of the center line?

[It is twice as much]

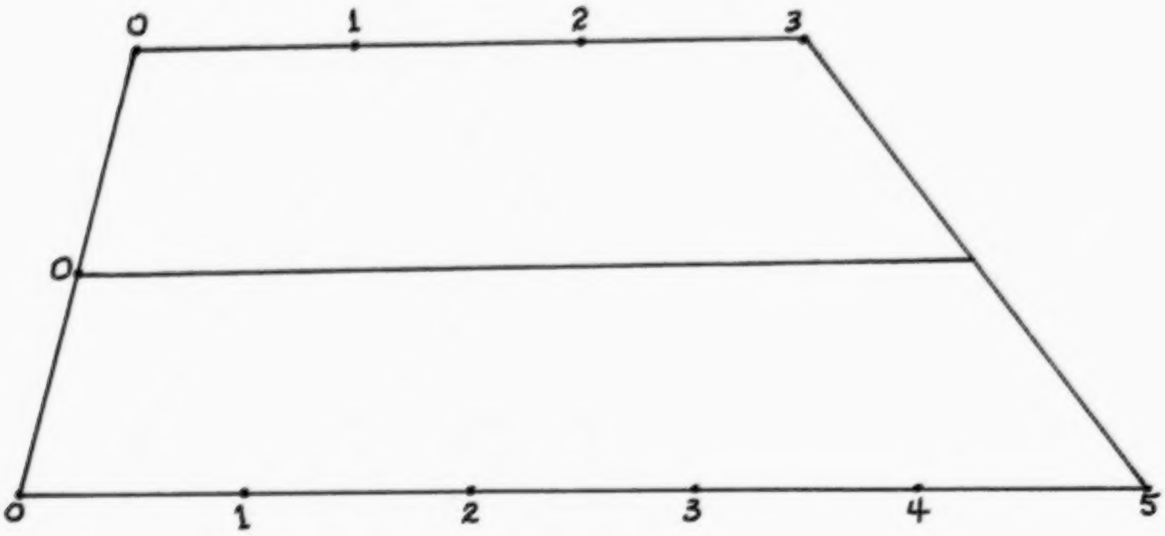
Application of Median of a Trapezoid to Nomogram

Nomogram: One of many mathematical devices used to find the sum of any two numbers.

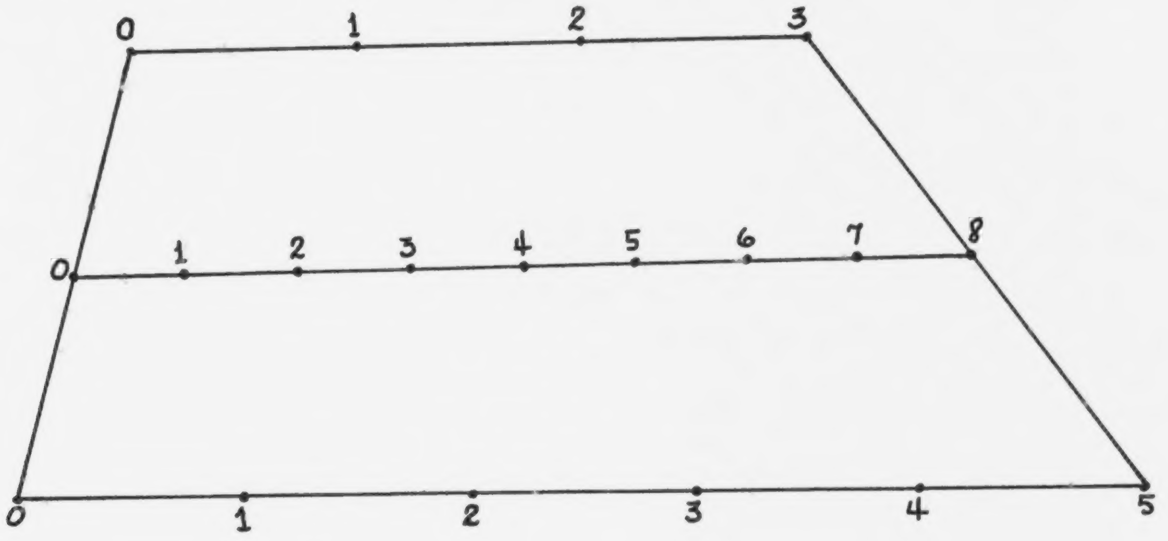
1. Have children mark off equal units on the upper and lower bases of their trapezoids and number the points as on a number line.
   For example:



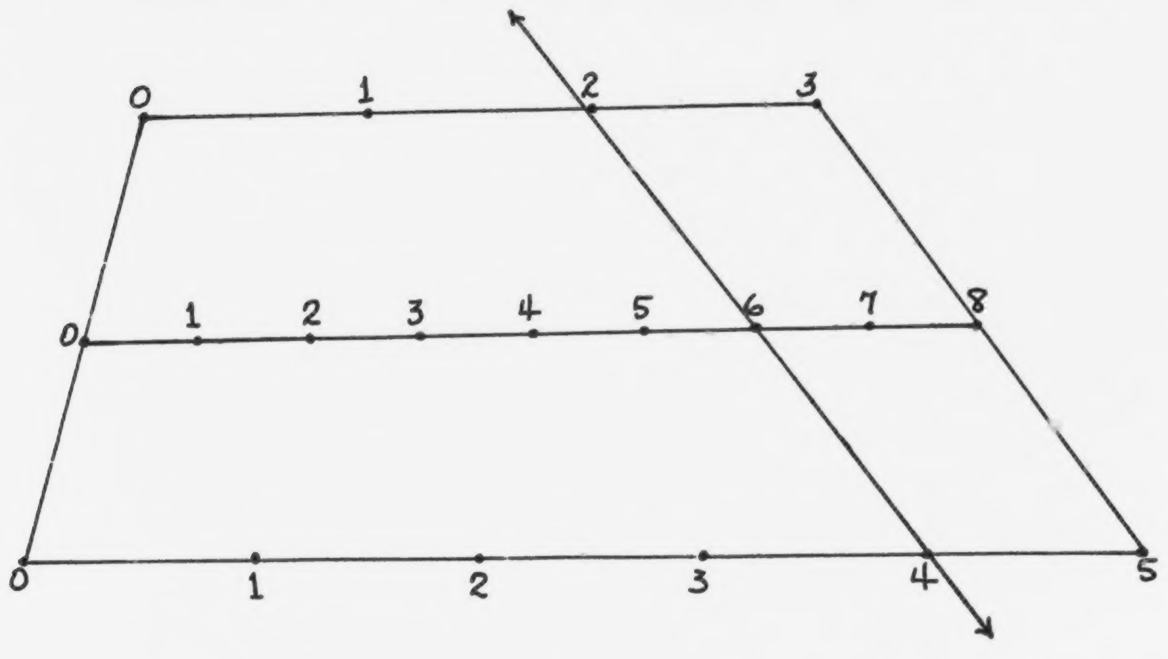
2. Have children find the midpoints and draw the center line.



3. Using a unit that is one half of the unit used on the other two sides, have children mark off the center line. They number the points as on a number line.


0
1
2
3
0
1
2
3
4
5
6
7
8
0
1
2
3
4
5

4. To add 2 and 4, the children locate 2 on the upper line and 4 on the lower line.
They place a ruler or straightedge so that the points 2 and 4 are on a line and are visible.

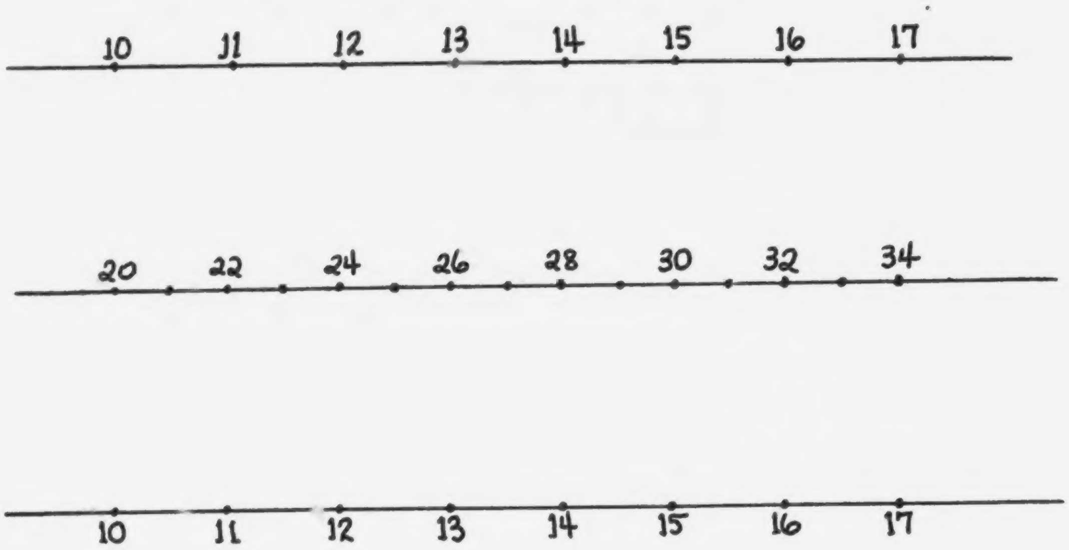

0
1
2
3
0
1
2
3
4
5
6
7
8
0
1
2
3
4
5

What point does the straightedge touch on the center line?   [6]

What is the relation of 6 to 2 and 4?

5. Tell children that this device is called a Nomogram.

6. Have them explore the nomogram with other numbers. This may be extended to include larger numbers. For example:


10
11
12
13
14
15
16
17
20
22
24
26
28
30
32
34
10
11
12
13
14
15
16
17

Ask children to:

Use a straightedge to show the sum of 10 and 12.

Show that 10 + 12 = 12 + 10
13 + 16 = 16 + 13   etc.

7. Have children discover how the Nomogram may be used for subtraction.

GEOMETRY AND MEASUREMENT

UNIT 28 - PERIMETER OF POLYGONS

Objectives: To develop concept of perimeter.
To help children derive formula for finding perimeter of rectangles.
To help children derive formula for finding perimeter of polygons of equal sides.

TEACHING SUGGESTIONS

1. Reinforce:

   Rectangles have 4 right angles and the opposite sides are parallel and have the same length.

   Squares have 4 right angles and the 4 sides are parallel and have the same length. Squares are special rectangles.

   Symbols for: foot or feet (') and inch or inches (").
   Terms: length, width, dimensions.

2. A rectangular picture, chart or map to be framed is presented by the teacher.

   Children estimate and then measure the length and width. Elicit from the children various ways of finding the amount of material needed to frame the picture or chart.

3. Introduce the concept Perimeter to mean the distance around. Emphasize that the perimeter of a rectangle means the distance around the four sides.

4. Ask children to estimate the length of picture wire needed to go around the four sides of framed picture. Have them open the wire and measure the length to verify their estimate.

Using the same method, find the perimeter of a table top, of a desk top, of other rectangular surfaces in the room. Compare the estimated perimeters with the measured perimeters.

5. Have children estimate and then measure:

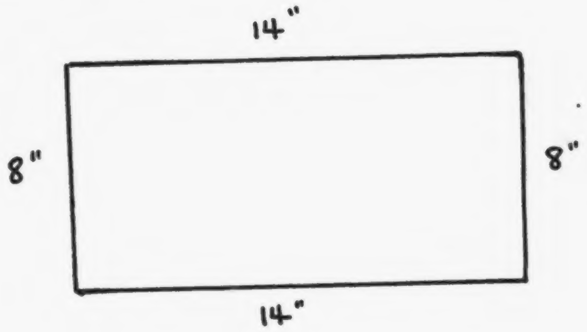
To find the amount of braid to go around a rectangular place mat.

To find the amount of leather binding needed to go around a rectangular foot stool.

To find the amount of crepe paper needed to go around a plant box.

To find the amount of lace needed for a table cloth edging.

Discuss to help children discover that:
The distance around a rectangle (perimeter) may be arrived at in various ways.



A. Adding: (two lengths and two widths, then combining them)

14 in. + 14 in. = 28"
8 in. + 8 in. = 16"
28 in. + 16 in. = 44" (answer)

B. Adding: (length and width and doubling sums)

14 in. + 8 in. = 22 in.
14 in. + 8 in. = 22 in.
22 in. + 22 in. = 44 in.

C. Adding: (1 length and 1 width, then multiplying by 2)

14 in. + 8 in. = 22 in.
2 x 22 in. = 44 in.

D. Multiplying: (length by 2, width by 2. Adding both amounts)

2×14 in. = 28 in.
2×8 in. = 16 in.
28 in. + 16 in. = 44 in.

Have children tell in their own words how to find perimeter of a rectangle.

Help children to arrive at formulas for Perimeter.

Discuss with children:

> If we let P stand for the Perimeter of the rectangle; "l" stand for its length; "w" stand for its width can you make a mathematical sentence to describe the Perimeter?

$$P = l + w + l + w$$
or
$$P = 2 \times (l + w)$$
or
$$P = l + l + w + w$$
or
$$P = 2l + 2w$$

6. Extend the above development to find perimeters of:

square - trimming a napkin or handkerchief
triangle - outlining a Christmas tree
hexagon - planning stop signs for safety week
other polygons - making stars, borders, laying out baseball diamonds, planning for carpeting stairs, etc.

Finding Perimeter of Polygons of Equal Sides

A polygon of equal sides is called an equilateral polygon.
Note: By "equal" here, we mean "equal in measure" or length.

Squares

Reinforce understanding of:

Right Angle: A right angle is an angle shaped like a square corner. Name some objects in the room that have right angles.

Square: A square is a quadrilateral having 4 sides of equal length and 4 right angles.

Discuss:

A square as a special type of rectangle.
Compare a square with other rectangles.
When is a rectangle not a square?
Is every rectangle a square? [No] Why?
Is every square a rectangle? [Yes] Why?

Problem: A handkerchief measures 8 inches on each side. How much lace is needed for trim around the handkerchief?

Discuss various ways of finding the perimeter.

$8 + 8 + 8 + 8 = n$ $\qquad$ $4 \times 8 = n$

Children find perimeters of other squares.

They arrive at the formula for finding the perimeter of a square.

4 times the number of units of length of a side = the perimeter of a square.

$p = 4 \times s$ or $p = 4s$

Equilateral Triangles, Other Equilateral Polygons

Discuss:

Equilateral Triangle, Rhombus, Pentagon, Hexagon, etc.

Children find perimeter of each type in a variety of ways.

They explain how they would arrive at the formula for finding the perimeter of equilateral polygons.

If "p" represents perimeter, and "s" represents the length of a side, then the perimeter of a:

Triangle is $p = 3 \times s$ or $p = 3s$
Square is $p = 4 \times s$ or $p = 4s$
Pentagon is $p = 5 \times s$ or $p = 5s$
Hexagon is $p = 6 \times s$ or $p = 6s$
Octogon is $p = 8 \times s$ or $p = 8s$ etc.

They arrive at the generalization that when lengths of the sides of a polygon are equal, the perimeter is found by multiplying the length of a side (s) by the number of sides (n).

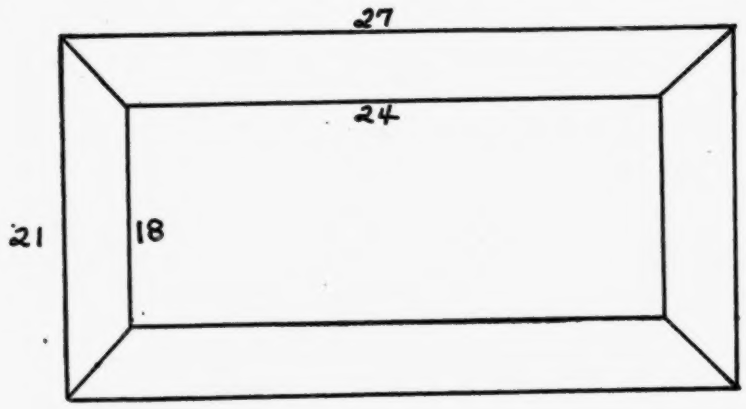
$$p = ns$$

Children complete the following and explain:

Perimeter of equilateral triangle = ⊓ times length of 1 side [3]
Perimeter of equilateral octogon = ⊓ times length of 1 side [8]
Perimeter of equilateral pentagon = ⊓ times length of 1 side [5]

EVALUATION and / or PRACTICE
SUGGESTED EXERCISES

1. Determine the amount of material needed to make a picture frame as shown below.



[21" + 27" + 21" + 27" = 96"]

2. The shape of the baseball diamond in our school yard is square. If one side is 80 ft., about how far does Bob run if he hits a home run? Use formula.

3. Solve the open sentence $P = 3s$ where $s = 10$. Tell what kind of polygon is involved.

4. How much fence is needed to enclose a triangular garden each of whose sides is 7'6".

5. Additional practice exercises in finding perimeter are to be found in textbooks.

SETS; NUMBER; NUMERATION

UNIT 29 - THE BASE-TEN SYSTEM OF NUMERATION EXTENDED

NOTE TO TEACHER

Relate large numbers to children's possible experiences, e.g.,

Seating capacities in various arenas:

Madison Square Garden	15,000	(circus)
	20,000	(boxing)
Yankee Stadium	68,000	
Soldiers' Field	100,000	(seen on television perhaps)
Phila. Municipal Stadium	110,000	

As an aid in reading and understanding large numbers, commas are used to set off groups of three digits, beginning at the right. Each of these groups of digits is called a period. The first period starting at the right, is called the units' period because the number expressed by its digits taken together is that many ones. The second period of a number is called the thousands' period because its digits express the number of thousands.

Objective: To extend understanding of Base Ten System to: hundred thousands; millions

TEACHING SUGGESTIONS

Test for understanding of the Base-Ten System of Numeration through 10,000; place value; reading and writing numbers.

Extend Understanding of Base Ten System to Hundred Thousands

1. Begin with 10,000 - 20,000 and gradually extend the system. Reinforce the fact that each place is ten times the value of the place to its right:

10,000 = 10 thousands;
100,000 = 10 ten thousands; also 100,000 = 100 thousands

Position determines the value of each digit, e.g.,

44,444

4 ten thousands = 40,000
4 thousands = 4,000
4 hundreds = 400
4 tens = 40
4 ones = 4
44,444

62,518

6 ten thousands = 60,000
2 thousands = 2,000
5 hundreds = 500
1 tens = 10
8 ones = 8
62,518

2. Have children rewrite the numerals in expanded notation.

44,444 = 40,000 + 4,000 + _____

62,518 = 60,000 + n + ☐ + 8

etc.

3. Introduce the period chart.

Record numerals: 142560 142,560

Ask children:

Which numeral is easier to read? Why?
Read that numeral.
Where is the comma placed?

Tell children that each group of 3 digits is called a period.

Present a period chart.

Thousands Period			Units Period		
Hundreds	Tens	Ones	Hundreds	Tens	Ones

Discuss: Ones, Tens, Hundreds in the Thousands Period.
Children note that in reading a number such as 41,235, the 41 is stated first, then the period in which it falls (thousands), then the 235 is stated.

4. Present numerals such as the following:

4
26
523
6,328
28,253
159,417

Have the children read each numeral and identify the periods into which each digit falls. For example: the 6 in 6,328 falls in the thousands period and the 328 in the units period.

Have them note the units period is not stated when reading numerals.

5. Present numerals such as the following:

2075 468506 8057
35240 507436 50289

Direct children to:

Copy these numerals and point off the periods.
Note: To place commas correctly children start at the right (from ones place) and go to the left.

Read the numbers aloud.

Tell or write the meaning of the digit 5 in each of the numbers.

Tell or write the meaning of zero in each of the numbers.

6. Dictate numbers.

Children write the number that is:

100 greater than each of these dictated numbers
1,000 greater than each of these dictated numbers
10,000 greater than each of these dictated numbers
100,000 greater than each of these dictated numbers

Counting

Indicate to the children how far to continue the sequence.
Examples follow:

Counting Forward

10,000, 20,000, 30,000, ___, ___, ___,
21,100, 22,100, 23,100, ___, ___, ___,
25,100, 25,200, 25,300, ___, ___, ___,
30,100, 30,200, 30,300, ___, ___, ___,
18,600, 18,700, 18,800, ___, ___, ___,

23, 46, 69, ___, ___, ___,
373, 381, 389, ___, ___, ___,
414, 428, 442, ___, ___, ___,
120, 240, 360, ___, ___, ___,
102, 204, 408, ___, ___, ___,

600 1200 1800 ___ ___ ___
406 812 1218 ___ ___ ___

11,000 21,000 31,000 ___ ___ ___
14,000 14,500 15,000 ___ ___ ___
100,000 200,000 300,000 ___ ___ ___

Counting Backward

60,000, 50,000, 40,000, ___, ___, ___,
26,100, 25,100, 24,100, ___, ___, ___,
25,600, 25,500, 25,400, ___, ___, ___,
30,600, 30,500, 30,400, ___, ___, ___,
19,400, 19,300, 19,200, ___, ___, ___,

252, 244, 236, ___, ___, ___,
121, 115, 109, ___, ___, ___,
96, 48, 24, ___, ___, ___,
720, 610, 500, ___, ___, ___,
128, 64, 32, ___, ___, ___,

520 480 440 ___ ___ ___
3500 3000 2500 ___ ___ ___

61,000 51,000 41,000 ___ ___ ___
16,500 16,000 15,500 ___ ___ ___
600,000 500,000 400,000 ___ ___ ___

Money

1. Test for understanding of place value and for ability to read, write, make change, regroup, etc., through $99.99.

2. Extend development through $999.99. Include:

    Place Value
    Regrouping - as for $130.00 - 129 dollars 10 dimes 0 pennies
    129 dollars 9 dimes 10 pennies
    Reading and Writing - regular and irregular columns
    Numbers in Series
    Change Making

Develop Understanding of Base Ten System to Millions

1. Test for understanding of our system of numeration through 6 places; place value; reading and writing of numerals; numbers in series.

    How many digits are used in our decimal system of numeration? What are they?

    Use numerals to express the following: three hundred five thousand three; seventy two thousand one hundred fifty six; ten thousand one.

Use words to express the following: 463,780; 25,021.

Write the numeral that comes after each of the following:

1,099 1,999 10,099

Starting with units place, name the next five places to the left.

State the place value and numerical value of the 4 in each of the following numerals:

411 [hundreds place; 400]
34
2,140
14,115
45,231

What place is the zero holding in each of the following?

40,728 207,354 629,072

Fill in the spaces:

Digit	Place	Numeral
5	Thousands	___[5000]
4	Thousands and Tens	___[4040]
6	Hundred-thousands and Thousands	___[606,000]
3	Ten Thousands and Units	___[30,003]

2. Record: 999,999

Ask children:

What number comes next? [1 million]

Write 1 million as a numeral. [1,000,000]

How many hundred thousands are in 1 million? [10 hundred thousands]

How many thousands are in 1 million? [1000 thousands]

3. Use a place value chart if necessary.

Millions	Hundred Th.	Ten Th.	Thousand	Hundreds	Tens	Ones

Record numerals on place value chart.

Discuss relationships between:

Ten Thousand and One Thousand
One Hundred Thousand and Ten Thousand
One Million and One Hundred Thousand

4. Write a sentence such as the following:

In 1958, the population of New York City was approximately 7891900.

Ask children:

How can the numeral be written so that it would be easier to read? [Leave spaces; use commas; mark off periods]

How are periods named? [Ones or units period; thousands period; millions period]

How are the places in each period named? [Hundreds, tens, ones]

Use a chart as children explain.

Millions			Thousands			Units		
H	T	O	H	T	O	H	T	O

Which period is not named when a numeral is read? [Units or ones]

5. Have children read the following numerals:

62,471,038 2,596,204 86,003,001

Why is the period farthest to the left the only period that may have one or two digits in it rather than three?

EVALUATION and / or PRACTICE
SUGGESTED EXERCISES

1. Write numerals for the following. Use commas to set off periods.

Fifty thousand, fifty.

Four hundred sixty-one thousand, seven hundred sixty eight.

Two million, two hundred.

2. Encircle the numerals where commas are placed incorrectly. Rewrite each to make it correct.

5,371,560 293,45 53,74,357 4,29,543,297

3. Place digits for 5073806 in proper columns. Then read the numeral.

[5 million, 73 thousand, 8 hundred six]

Millions	Hundred Th.	Ten Th.	Thousands	Hundreds	Tens	Ones
[5]	[0]	[7]	[3]	[8]	[0]	[6]

4. Rewrite the numeral inserting commas to mark off periods. [5,073,806]

5. Rewrite the following indicating periods.

5550500 5050005 55005550 555500050

6. Mark the following True or False. If false, make the statement true.

   a. The largest number that can be expressed by a Hindu-Arabic numeral of eight digits is one less than 10,000,000.

   b. There could be some counting numbers that are so large that new symbols would be needed to express them.

OPERATIONS

UNIT 30 - SET OF WHOLE NUMBERS: ADDITION AND SUBTRACTION; FACTS; HORIZONTAL FORMAT; PROPERTIES APPLIED

Objectives: To help children develop skill in adding and subtracting whole numbers.

To make sure children can apply the Commutative and Associative Properties of Addition.

To make sure that children understand that subtracting a number is the inverse operation of adding that same number.

TEACHING SUGGESTIONS

1. Discuss addition as an operation.
   Compare adding a number to a given number, with combining the elements of a set to another set.

2. Test and / or drill where necessary to maintain automatic response to addition and subtraction facts.

   Develop skill in applying addition and subtraction facts to numbers through thousands. Use the horizontal format. Have children record sums and remainders only.

3. Present open sentences to evaluate children's understanding of mathematical properties

   For example,

   a. Children complete the following and explain

      $36 + 53 = 53 + \square$ Why?
      $36 + 53 = 36 + (\square + 3)$ Why?
      $36 + 53 = (30 + \square) + (6 + \triangle)$ Why?

      $89 - 53$ is the inverse operation of $36 + \square$ Why?
      $36 + \square$ is the inverse operation of $89 - \square$ Why?

b. Children find the missing addend "mentally"

$\square + 42 = 69$
$83 = 47 + \square$
$64 + \square = 113$
$122 = \square + 85$

$\square + 35 = 161$
$63 + \square = 281$
$485 = 129 + \square$
$\square + 338 = 565$

$\square + \$56 = \$87$
$\$20 = \square + \$75$
etc.

c. Ask children to find the missing term.

$\square - 23 = 75$
$57 = \square - 35$

$\square - 65 = 87$
$\square - 374 = 233$

$\square - \$24 = \$56$
$\$105 = \square - \$57$

d. Applying the Associative Property of Addition in "Mental Computation"

Ask children to solve the equations below applying the Associative Property.

For example,

$$\begin{aligned} 68 + 43 &= 68 + (40 + 3) \\ &= (68 + 40) + 3 \end{aligned}$$

or

$$\begin{aligned} 68 + 43 &= (60 + 8) + (40 + 3) \\ &= (60 + 40) + (8 + 3) \end{aligned}$$

$32 + 95 = n$ $\qquad 324 + 59 = n$ $\qquad 276 + 138 = n$, etc.

They explain their reasoning in each case.

4. Suggested exercises for adding and subtracting "mentally"

a.
$360 + 80 = n$
$364 + 80 = n$
$364 + 85 = n$
$364 + 88 = n$

$240 + 130 = n$
$246 + 133 = n$
$246 + 138 = n$
$431 + 259 = n$

b. A similar series for subtraction should be established. For example,

$210 - 50 = n$
$210 - 54 = n$
$218 - 54 = n$
$218 - 59 = n$

$240 - 120 = n$
$376 - 130 = n$
$376 - 134 = n$
$376 - 137 = n$

Both series should be extended to include numbers in the thousands.

Drill should first be given in adding tens to thousands, then hundreds to thousands, etc.

c. Suggested exercises involving dollars and cents

Adding dollars

$\$4.16 + \$2 = n$
$\$23.89 + \$5 = n$
$\$68.19 + \$4 = n$

Adding dollars and cents

$\$4.20 + \$4.20 = n$
$\$3.46 + \$2.10 = n$
$\$14.61 + \$3.30 = n$
$\$37.43 + \$4.20 = n$
$\$4.61 + \$4.61 = n$

Subtracting dollars

$\$6.16 - \$2 = n$
$\$28.89 - \$5 = n$
$\$72.19 - \$4 = n$

Subtracting dollars and cents

$\$5.56 - \$2.10 = n$
$\$8.40 - \$4.20 = n$
$\$17.91 - \$3.30 = n$
$\$41.63 - \$4.20 = n$
$\$9.22 - \$4.60 = n$

EVALUATION and / or PRACTICE

SUGGESTED EXERCISES

1. How many books were borrowed by 5th grade pupils on Monday? Where will this number be placed in the table?

NUMBER OF BOOKS BORROWED BY 5TH GRADES FROM SCHOOL LIBRARY

	Mon.	Tues.	Wed.	Thurs.	Fri.	Total
Class 5-1	9	8	16	12	29	
Class 5-2	6	8	15	10	34	
Class 5-3	3	9	12	13	25	
Class 5-4	14	8	6	8	32	
Total						

How many books were borrowed by 5th grade pupils on Tuesday? on Wednesday? on Thursday? on Friday?

From your answers to 1 and 2, find the total number of books borrowed by 5th grade pupils during the week.

How many books did pupils from class 5-1 borrow during the week?

How many books did pupils from class 5-2 borrow during the week?

How many books did pupils from class 5-3 borrow during the week?

How many books did pupils from class 5-4 borrow during the week? etc.

2. Replace each placeholder with a numeral that will make the statement true.

$54 + 39 = 39 + \square = n$
$54 + 39 = 54 + \square + \Delta = n$
$54 + 39 = 54 + \square - 1 = n$
$54 + 39 = 50 + 40 + 4 - \square = n$

3. Without adding tell how you know that each of the following is a true sentence.

$3679 + 2000 = 2000 + 3679$

$153 + (10 + 6) = (153 + 10) + 6$

$968 = 968 + 0$

4. Further practice exercises may be found in textbooks and other printed material.

OPERATIONS

UNIT 31 - ADDITION AND SUBTRACTION OF WHOLE NUMBERS; **VERTICAL FORMAT;** PROPERTIES APPLIED

Objective: To maintain skill in:
Renaming numbers
Adding and subtracting whole numbers in vertical format

TEACHING SUGGESTIONS

1. Test children's ability to solve addition and subtraction exercises involving numbers as suggested in Unit 23.

2. Continue to develop or to provide practice in addition with 3-place numerals, sums in the thousands, and with numerals representing quantities of money, addends through $9.99.

   Present problem situations.

   Have children estimate, compute, then compare solution with estimate and check.

   Some suggested exercises:

436	568	39	$2.24	$3.15	
325	207	256	7.65	4.59	
$\underline{547}$	472	947	8.36	7.28	
	$\underline{93}$	$\underline{81}$	$\underline{.59}$	$\underline{1.50}$	etc.

3. Extend renaming numbers to include thousands.
   For example, 4030 may be renamed as:

   a. 3 thousands + 10 hundreds + 3 tens + 0 ones; or
   3 thousands + 10 hundreds + 2 tens + 10 ones; or
   3 thousands + 9 hundreds + 12 tens + 10 ones; or
   39 hundreds + 13 tens + 0 ones

b. Direct children to:
Complete exercises like the following

4321 = ...thousands + ...hundreds + ...tens + . . . ones
4321 = ...hundreds + ...ones
4321 = ...tens + ...ones

54321 = ...ten-thousands + . . . thousands + . . . hundreds + . . . tens + . . . ones
54321 = ...thousands + ... tens + ... ones

c. Record the following in expanded notation

39 hundreds + 13 tens + 0 ones [3900 + 130 + 0 = 4030]
43 hundreds + 3 ones
24 thousands + 7 tens + 8 ones

4. Provide practice in renaming numbers.
Ask children to complete the following:

2526 = 2000 + 500 + 20 + □
4000 = 3000 + □ + 90 + 10
3802 = □ + 1800 + 0 + 2
6000 = □ hundreds + 10 tens + 0 ones

5. Ask children to estimate sums and remainders, then complete and compare.

Sums:

Continue to emphasize estimating prior to written computation

761 Add hundreds only. Estimate: 1400 + **or**
376 Add hundreds, then tens: (1400 + 180). Estimate: 1580 **or**
$\underline{459}$ Add the first two digits: 76 (tens) + 37 (tens) + 45 (tens). Estimate: 1580 (158 tens)

3671 Add thousands only. **Estimate:** 6000 + **or**
2298 Add thousands, then hundreds: 6000 + 1300. **Estimate: 7300 + or**
1538 Add the first two digits: 36 (hundreds) + 22 (hundreds) + 15 (hundreds). Estimate: 73 hundred or 7300

Remainders:
6279 Subtract thousands only. Estimate: about 4000.
$\underline{-\ 2930}$ Subtract 3000 from 6000. Estimate: about 3000.
Subtract 3000 from 6279. Estimate: about 3279.
Subtract 29 (hundred) from 62 (hundred). Estimate: 3300 (33 hundred).

From time to time ask children at varying levels of ability to tell how they arrived at their estimates. Mature children should be encouraged to use mature methods of estimation. Children unable to make reasonable estimates should use smaller numbers.

6. Continue to develop skill in addition and subtraction.

Addition with 4-place numerals - 2, 3, 4 addends - sums in the thousands and ten thousands - first 1 exchange, then 2, 3, 4 exchanges; with numerals representing quantities of money - no addend larger than $99.99 - first with 1 exchange, then 2, 3, 4 exchanges.

Some suggested exercises.

4118	298	2814	2376	4625	$34.98	$52.28
2457	3665	2498	8924	537	68.50	19.79
	2033	1573		3869		8.54
		2865		5386		

Extend addition development to include:

5-place numerals - 2, 3, 4 addends - sums in the ten thousands - 1, 2, 3, 4 exchanges

Numerals representing quantities of money - maximum sum $999.99 - 1, 2, 3, 4 exchanges

Some suggested exercises

25,694	$57.96	16,385	
13,835	43.85	42,031	
3,610	25.19	31,400	
20,360		6,193	etc.

Subtraction

Continue to develop or provide practice in subtracting from numbers through 9999; through $99.99 - 1 and 2 exchanges.

Exchanges should include as many as 2 zeros.

Extend development to include subtracting from:

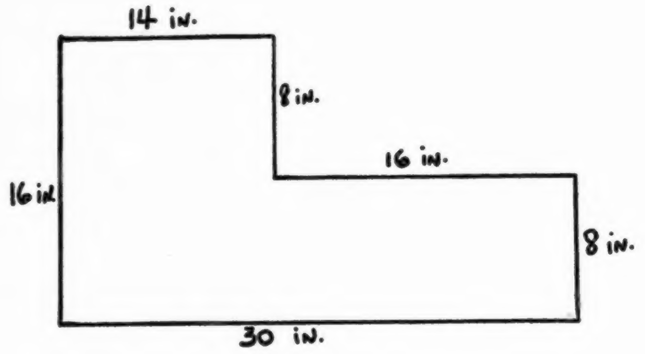
Numbers through 99,999; 1 and 2 exchanges; a maximum of 2 zeros.
Numbers through $999.99; 1 and 2 exchanges

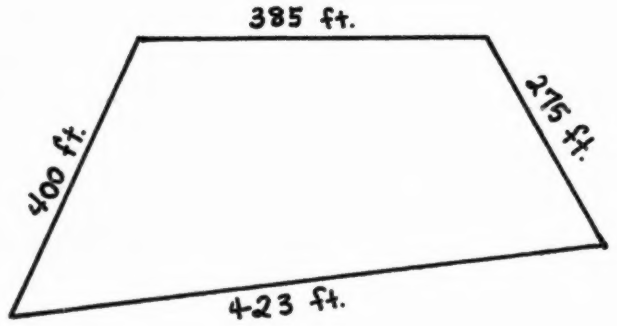
Some suggested exercises

58, 780	15, 759	$600.79
- 42, 352	- 9, 241	- 124.26

EVALUATION and / or PRACTICE
SUGGESTED EXERCISES

1. Find perimeters of polygons by adding.


14 in.
8 in.
16 in.
16 in.
8 in.
30 in.


385 ft.
400 ft.
275 ft.
423 ft.

3. A rectangle measures 8 inches on each of two sides. A third side measures 6 inches. What is the length of the fourth side?

4. Provide for maintenance of computational skills. Refer to texts.

   Children continue to estimate sums and remainders. Verify sums and remainders.

   Select suitable verbal problems from textbooks and from other curriculum areas.

SETS; NUMBER; NUMERATION

UNIT 32 - SET OF FRACTIONAL NUMBERS: SIXTHS; RELATED TO HALVES AND THIRDS; CONCEPTS; COUNTING.

Objectives: To develop understanding of sixths.

To extend understanding of relationships among fractions including inequalities.

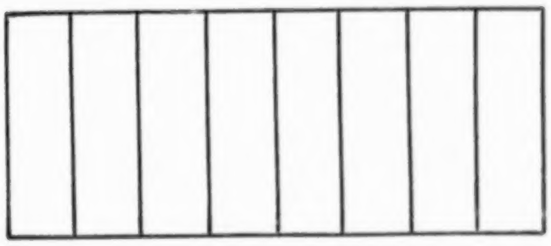
To develop understanding of whole numbers named as fractions.

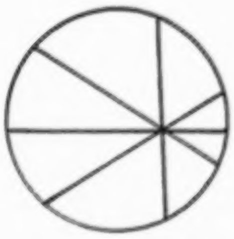
TEACHING SUGGESTIONS

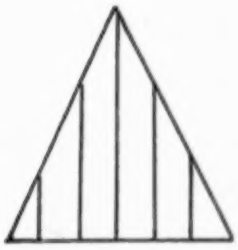
1. Continue to use number lines to reinforce relationships among halves, halves and fourths, halves, fourths and eighths, fourths and eighths.

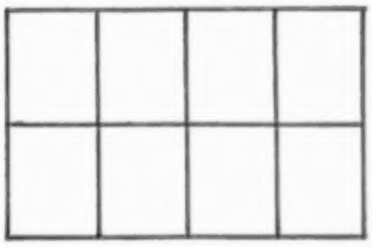
   Suggested exercises for evaluation

   a. Check the diagrams that show which region has been divided into eighths.

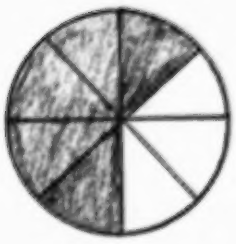


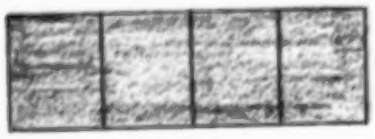


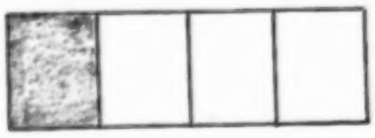


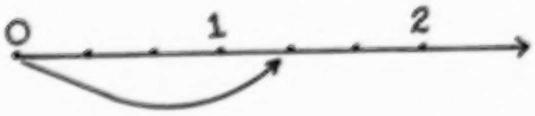


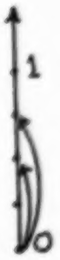
b. Record symbols to indicate the size of the shaded and indicated parts of the diagrams below.











c. Write the fraction that is twice as large as $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{3}$

Arrange in order of size: $\frac{1}{4}$, $\frac{1}{8}$, $\frac{5}{8}$, $\frac{1}{3}$, $\frac{4}{3}$, $\frac{1}{3}$

d. Circle the fractions in each line which have the same value as the first fraction:

$\frac{1}{2}$ - $\frac{2}{3}$ $\frac{2}{4}$ $\frac{4}{8}$

$\frac{1}{4}$ - $\frac{2}{8}$ $\frac{4}{8}$ $\frac{1}{2}$

$\frac{4}{8}$ - $\frac{3}{2}$ $\frac{2}{3}$ $\frac{2}{4}$

e. Show the following on a number line

$\frac{1}{2}$ is the same point as $\frac{2}{4}$ or $\frac{4}{8}$ on the number line

$\frac{6}{8}$ is equivalent to $\frac{3}{4}$ or $\left(\frac{1}{2} + \frac{3}{4}\right)$

$\frac{3}{3}$ is equivalent to 1 whole

$\frac{6}{3}$ is equivalent to 2 wholes

$\frac{7}{3}$ is equivalent to 2 wholes and 1 third $\left(2\frac{1}{3}\right)$, etc.

f. Circle the larger (or smaller) of the following pairs:

$\frac{1}{3}$ $\frac{1}{4}$

$\frac{3}{8}$ $\frac{1}{4}$

$\frac{3}{2}$ $\frac{4}{4}$

$\frac{6}{4}$ $1\frac{1}{4}$

$2\frac{1}{8}$ $\frac{9}{4}$

etc.

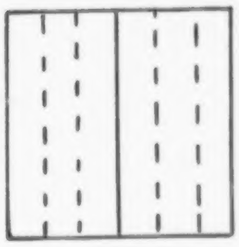
2. Develop concept of sixths

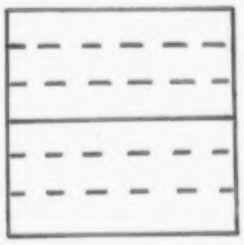
Problem: You have several sheets of paper of the same size. Fold each into 6 equal parts in different ways.

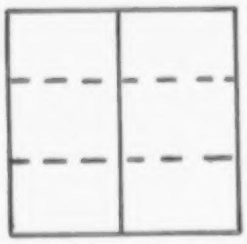
Ask children to describe in their own words how they derived sixths.

Some explanations might be

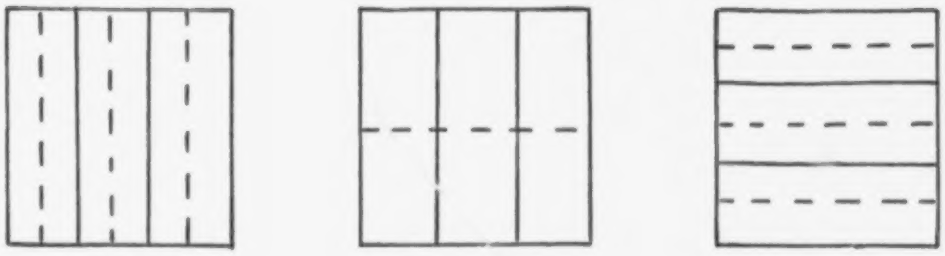
a. Finding halves, then dividing each half into 3 equal parts.







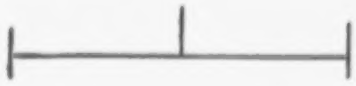
b. Finding thirds, then dividing each third in half



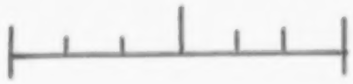
Ask children to identify the name of the part and to tell how they discovered that each part is 1 sixth of its original unit despite difference in shapes.

3. Use number lines to extend concepts of sixths in relation to halves and thirds.

   Have children draw a line segment to represent a unit. They divide the unit into 2 equal parts.



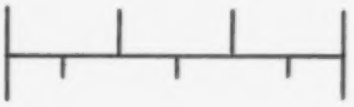
Children suggest ways to indicate sixths on the line segment divided into 2 equal parts.



Discuss dividing the line segment into halves first, then each half into 3 equal parts.

Ask children to draw a line segment to represent one unit. They divide the unit into 3 equal parts.

Children suggest ways to indicate sixths on the line segment divided into 3 equal parts.



Discuss dividing the line into thirds first, then each third into 2 equal parts.

Discuss the following comparisons.

Sixths of the same or of equal wholes are the same size

One sixth derives its name because it is one of 6 equal parts of a whole

There are 6 sixths in a whole

Discuss the following relationships. Use number lines. How are sixths related to thirds and to halves?

$\frac{1}{6}$ is half as large as $\frac{1}{3}$, or $\frac{1}{6} = \frac{1}{2}$ of $\frac{1}{3}$

$\frac{1}{3}$ is twice the size of $\frac{1}{6}$, or $\frac{1}{3} = 2$ one sixths

$\frac{1}{6}$ is $\frac{1}{3}$ the size of $\frac{1}{2}$, or $\frac{1}{6} = \frac{1}{3}$ of $\frac{1}{2}$

$\frac{1}{2}$ is 3 times as large as $\frac{1}{6}$, or $\frac{1}{2} = \frac{1}{6}$ taken 3 times or 3 one sixths

1 whole is 6 times as large as $\frac{1}{6}$, or $1 = \frac{1}{6}$ taken 6 times or 6 one sixths

Have children complete the statements below

$\frac{1}{6} = \square$ of $\frac{1}{2}$

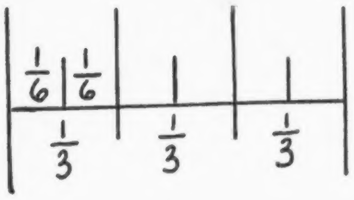
$\frac{1}{2}$ is $\square$ times $\frac{1}{6}$

$\frac{1}{3}$ is $\square$ times $\frac{1}{6}$

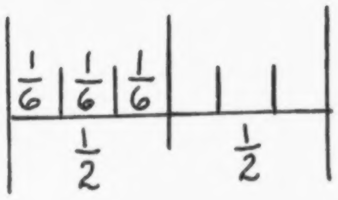
1 is $\square$ times $\frac{1}{6}$

4. Emphasize equivalencies between:

Sixths and Thirds


$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{3}$

Sixths and Halves


$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{2}$
$\frac{1}{2}$

Have children complete the following open sentences.

$\frac{2}{6} = \frac{\square}{3}$

$\frac{3}{6} = \frac{1}{3} + \frac{\square}{6}$

$\frac{4}{6} = \frac{\square}{3}$

$\frac{5}{6} = \frac{2}{3} + n$

$\frac{3}{6} = \frac{\square}{2}$

$\frac{4}{6} = \frac{1}{2} + n$

$\frac{5}{6} = \frac{1}{2} + n$

$\frac{5}{6} = \frac{1}{2} + \frac{?}{3}$

$\frac{7}{6} = 1 + n$

$\frac{8}{6} = 1 + \frac{n}{6}$

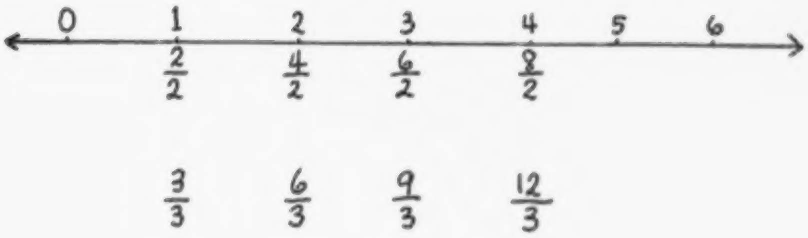
$\frac{8}{6} = 1 + \frac{n}{3}$

Ask children to rename one of the fractions in each of the following sets to make the denominators the same.

$\frac{1}{2}, \frac{5}{6};$ $\left[\frac{3}{6}\right]$ $\frac{2}{3}, \frac{2}{6};$ $\frac{4}{6}, \frac{2}{3};$ etc.

5. Extend concepts of equivalent fractions.

On the number line below, rename 1, 2, 3, 4, 5, 6 as halves; as thirds; as fourths; sixths; eighths.



Name other halves in the interval between 0 and 6. $\left[\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \text{etc.}\right]$

Other thirds in the interval between 0 and 6. etc.

What does 2 in the denominator tell us? 3? , 4? , etc.

6. Extend concepts of fractions to include denominators of one.

Ask children:

If 2 units of length are not divided into halves, thirds, fourths, etc. but are left whole, what denominator could be used to show this? [1]

What does the fraction $\frac{6}{1}$ indicate? Why?

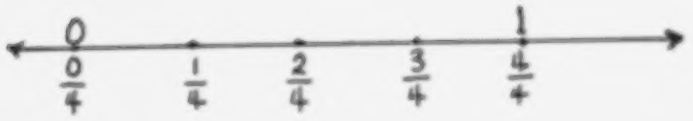
Direct children to:

Write 2 wholes as a fraction with a denominator of 1 $\left[\frac{2}{1}\right]$;

3 wholes $\left[\frac{3}{1}\right]$; 4 wholes $\left[\frac{4}{1}\right]$; 1 whole $\left[\frac{1}{1}\right]$; etc.

Draw another number line renaming 1, 2, 3, 4 in 5 different ways.

7. Use the number line to answer the questions below.



In the fraction $\frac{0}{4}$, what do the denominator and the numerator tell us?

If the line had been divided into halves, where would you record no halves? [at the zero point] How? $\left[\frac{0}{2}\right]$

What whole number does $\frac{0}{2}$ represent?

If the numerator of any fraction is zero, what whole number does that fraction represent? [0]

8. Compare with other fractions:

Children compare sixths with other fractions.

Extend understanding through use of circular materials, then number lines.

$\frac{1}{6} > \frac{1}{8}$ Why? $\frac{1}{6} < \frac{1}{4}$ Why?

$\frac{5}{6} > \frac{1}{2}$ Why? $\frac{3}{6} < \frac{2}{3}$ Why?   etc.

Ask children to write " > " or " < " between each group of fractional numerals below to make a true statement.

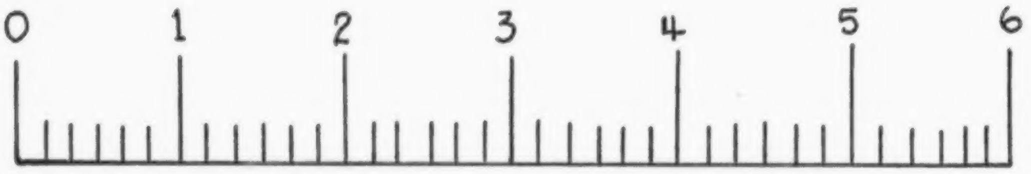
$\frac{2}{6} \square \frac{1}{4}$; $\frac{3}{4} \square \frac{3}{6}$; $\frac{3}{6} \square \frac{1}{8}$; $\frac{1}{2} \square \frac{5}{6}$; etc.

Draw number lines to show that each relation is correct.

9. Counting

Reinforce counting forward and backward by halves, fourths and eighths as suggested in Topic 15.

Ask children to use number lines to count forward and backward by thirds and sixths.



$\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3} \ldots$ **Later:** $\frac{1}{3}, \frac{2}{3}, 1, 1\frac{1}{3}, 1\frac{2}{3} \ldots$

$\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6} \ldots$ **Later:** $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6} \ldots$

$2\frac{1}{6}, 2\frac{2}{6}, 2\frac{3}{6}, 2\frac{4}{6} \ldots$ **Later:** $2\frac{1}{6}, 2\frac{1}{3}, 2\frac{1}{2}, 2\frac{2}{3} \ldots$

$3\frac{4}{6}, 4, 4\frac{2}{6}, 4\frac{4}{6}, 5 \ldots$ **Later** $3\frac{2}{3}, 4, 4\frac{1}{3}, 4\frac{2}{3}, 5 \ldots$

Have children start at any point:

Count backward by thirds and sixths
Count forward and backward by groups of 1 half, 1 third, 1 fourth, 1 sixth, etc. For example:

$0, \frac{2}{3}, \frac{4}{3}, \frac{6}{3}, \frac{8}{3}, \frac{10}{3}, \frac{12}{3}, \ldots$

$0, \frac{2}{3}, 1\frac{1}{3}, 2, 2\frac{2}{3}, 3\frac{1}{3}, 4, \ldots$

10. Terminology

Provide for a growing mathematical vocabulary by introducing and using meaningfully the following terms: fraction, numerator, denominator, fractional numeral, whole number and fraction.

Numerator and Denominator

3 (numerator) - the number of parts considered

4 (denominator) - the number of equal parts into which the whole has been divided.

Whole Number indicates a number of units. 1 7 253 etc.

SETS; NUMBER; NUMERATION

UNIT 33 - SET OF FRACTIONAL NUMBERS: TWELFTHS; NINTHS; SEVENTHS

Objective: To develop the understanding of concept of twelfths, ninths and sevenths.

TEACHING SUGGESTIONS

Twelfths

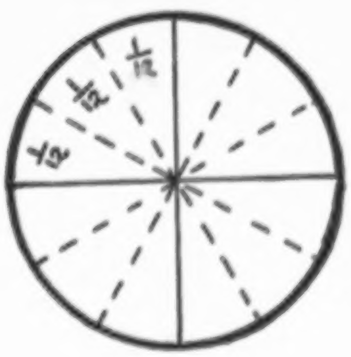
1. Relate to halves and fourths; halves and sixths; thirds and sixths. Use circle and number line diagrams.

2. Suggested problem:

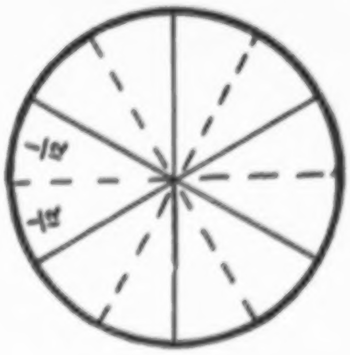
   Mother bought a pie large enough for 12 people. How can she cut it so that each person will receive the same amount?

   Children may draw circles or line segments. By experimenting they will discover various ways to arrive at twelfths. Why are these equal parts called twelfths?

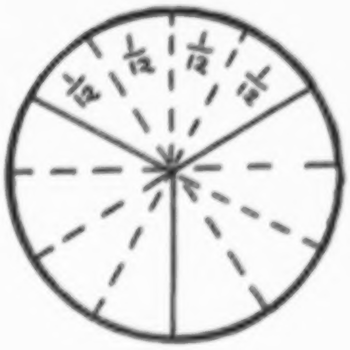
Divide into halves,
then fourths,
then twelfths



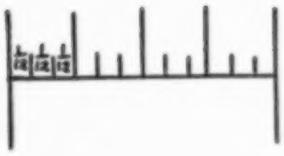
Divide into halves,
then sixths,
then twelfths



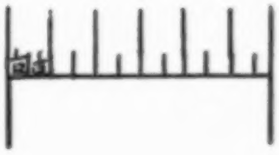
Divide into thirds,
then sixths,
then twelfths



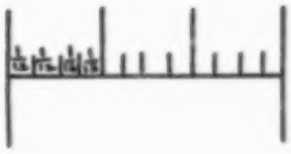
Each fourth is divided into 3 equal parts.



Each half is divided into 3 equal parts, then each sixth is divided into 2 equal parts.



Each third is divided into 2 equal parts, then each sixth is divided into 2 equal parts.



3. Suggested questions to ask:

How did we obtain:

$\frac{1}{12}$ from $\frac{1}{4}$? $\frac{1}{12}$ is what part of $\frac{1}{4}$? $\left[\frac{1}{12} = \frac{1}{3} \text{ of } \frac{1}{4}\right]$

$\frac{1}{12}$ from $\frac{1}{6}$? $\frac{1}{12}$ is what part of $\frac{1}{6}$? $\left[\frac{1}{12} = \frac{1}{2} \text{ of } \frac{1}{6}\right]$

$\frac{1}{12}$ from $\frac{1}{3}$? $\frac{1}{12}$ is what part of $\frac{1}{3}$? $\left[\frac{1}{12} = \frac{1}{4} \text{ of } \frac{1}{3}\right]$

$\frac{1}{12}$ from $\frac{1}{2}$? $\frac{1}{12}$ is what part of $\frac{1}{2}$? $\left[\frac{1}{12} = \frac{1}{6} \text{ of } \frac{1}{2}\right]$

After the pie is cut into 12 pieces:

What part of this pie would 6 pieces be? $\left[\frac{1}{2}\right]$ Why? $\left[\frac{6}{12} = \frac{1}{2}\right]$

What part of this pie would 4 pieces be? $\left[\frac{1}{3}\right]$ Why? $\left[\frac{4}{12} = \frac{1}{3}\right]$

What part of this pie would 3 pieces be? $\left[\frac{1}{4}\right]$ Why? $\left[\frac{3}{12} = \frac{1}{4}\right]$

What part of this pie would 2 pieces be? $\left[\frac{1}{6}\right]$ Why? $\left[\frac{2}{12} = \frac{1}{6}\right]$

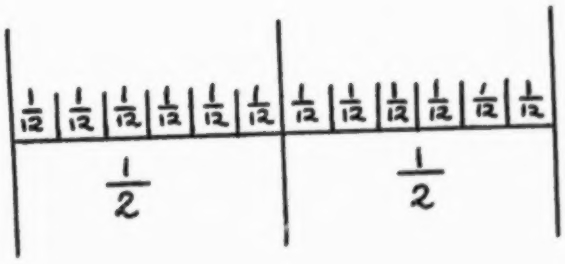
How many pieces are in $\frac{1}{4}$ of the pie? $[3]$ Why? $\left[\frac{1}{4} = \frac{3}{12}\right]$

How many pieces are in $\frac{1}{2}$ of the pie? $[6]$ Why? $\left[\frac{1}{2} = \frac{6}{12}\right]$

4. Equivalent Fractions

Use line diagrams to discover equivalents. Have children complete sentences.

Twelfths and Halves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$\frac{1}{2}$
$\frac{1}{2}$

$\frac{6}{12} = \square$

$\frac{12}{12} = \frac{\square}{2}$

$\frac{12}{12} = \square$

$\frac{6}{12} = \frac{1}{2}$

$\frac{7}{12} = \frac{1}{2} + \square$

$\frac{8}{12} = \frac{1}{2} + \frac{2}{\square}$

$\frac{9}{12} = \frac{1}{2} + \frac{\square}{12}$

$\frac{10}{12} = \square + \frac{4}{12}$

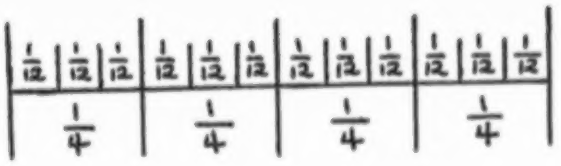
$\frac{11}{12} = \frac{1}{2} + \frac{5}{\square}$

$\frac{12}{12} = \frac{1}{2} + \frac{\square}{12}$

$\frac{12}{\square} = 1$

Since $\frac{6}{12} = \frac{1}{2}$ then $\frac{10}{12} = \frac{1}{2} + \frac{?}{12}$

Twelfths and Fourths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$\frac{1}{4}$
$\frac{1}{4}$
$\frac{1}{4}$
$\frac{1}{4}$

$\frac{3}{12} = \frac{\square}{4}$

$\frac{6}{12} = \frac{\square}{4}$

$\frac{9}{12} = \frac{\square}{4}$

$\frac{12}{12} = \frac{\square}{4}$

$1 = \frac{\square}{12}$

$\frac{4}{12} = \frac{1}{4} + \frac{\square}{12}$

$\frac{5}{12} = \frac{1}{4} + \frac{\square}{12}$

$\frac{7}{12} = \frac{2}{4} + \frac{\square}{12}$

$\frac{8}{12} = \frac{2}{4} + \frac{\square}{12}$

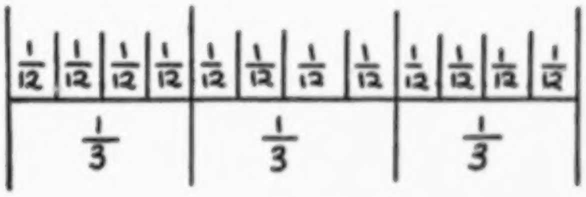
$\frac{10}{12} = \frac{3}{4} + \frac{\square}{12}$

$\frac{11}{12} = \frac{3}{4} + \frac{\square}{12}$

$\frac{12}{12} = \frac{3}{4} + \frac{\square}{12}$

Since $\frac{9}{12} = \frac{3}{4}$ then $\frac{11}{12} = \frac{3}{4} + ?$

Twelfths and Thirds



$\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$
$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$

$\frac{4}{12} = \frac{\square}{3}$

$\frac{8}{12} = \frac{\square}{3}$

$\frac{12}{12} = \frac{\square}{3}$

$\frac{5}{12} = \frac{1}{3} + \frac{\square}{12}$

$\frac{6}{12} = \frac{\square}{3} + \frac{2}{12}$

$\frac{7}{12} = \frac{1}{3} + \frac{\square}{12}$

$\frac{9}{12} = \frac{\square}{3} + \frac{1}{12}$

$\frac{10}{12} = \frac{2}{3} + \frac{\square}{12}$

$\frac{11}{12} = \frac{2}{3} + \frac{\square}{12}$

$\frac{12}{12} = \frac{\square}{3} + \frac{4}{12} = \frac{2}{3} + \frac{\square}{3}$

Since $\frac{4}{12} = \frac{1}{3}$ then $\frac{5}{12} = \frac{1}{3} + ?$

Twelfths and Sixths - Use Line Diagrams

$\frac{2}{12} = \frac{\square}{6}$

$\frac{4}{12} = \frac{\square}{6}$

$\frac{6}{12} = \frac{\square}{6}$

$\frac{8}{12} = \frac{\square}{6}$

$\frac{10}{12} = \frac{\square}{6}$

$\frac{12}{12} = \frac{\square}{6}$

$\frac{3}{12} = \frac{1}{6} + \frac{\square}{12}$

$\frac{5}{12} = \frac{\square}{6} + \frac{1}{12}$

$\frac{7}{12} = \frac{\square}{6} + \frac{1}{12}$

$\frac{9}{12} = \frac{\square}{6} + \frac{1}{12}$

$\frac{11}{12} = \frac{\square}{6} + \frac{1}{12}$

Since $\frac{4}{12} = \frac{2}{6}$ then $\frac{8}{12} = \frac{?}{6}$

Have children insert the correct numerator:

$\frac{1}{2} = \frac{\square}{12}$

$\frac{1}{3} = \frac{\square}{12}$

$\frac{1}{4} = \frac{\square}{12}$

$\frac{1}{6} = \frac{\square}{12}$

$\frac{12}{12} = \frac{\square}{2}$

$\frac{12}{12} = \frac{\square}{3}$

$\frac{12}{12} = \frac{\square}{4}$

$\frac{12}{12} = \frac{\square}{6}$

5. Comparisons and Relationships
Use number lines. Have children discover the following relationships:

$\frac{1}{12}$ is half as large as $\frac{1}{6}$, or $\frac{1}{12} = \frac{1}{2}$ of $\frac{1}{6}$

$\frac{1}{6}$ is twice as large as $\frac{1}{12}$, or $\frac{1}{6} =$ 2 one-twelfths

$\frac{1}{12}$ is 1 fourth as large as $\frac{1}{3}$, or $\frac{1}{12} = \frac{1}{4}$ of $\frac{1}{3}$

$\frac{1}{3}$ is 4 times as large as $\frac{1}{12}$, or $\frac{1}{3} =$ 4 one-twelfths or $4 \times \frac{1}{12}$

$\frac{1}{12}$ is 1 third as large as $\frac{1}{4}$, or $\frac{1}{12} = \frac{1}{3}$ of $\frac{1}{4}$

$\frac{1}{4}$ is three times as large as $\frac{1}{12}$, or $\frac{1}{4} =$ 3 one-twelfths or $3 \times \frac{1}{12}$

$\frac{1}{12}$ is 1 sixth as large as $\frac{1}{2}$, or $\frac{1}{12} = \frac{1}{6}$ of $\frac{1}{2}$

$\frac{1}{2}$ is 6 times as large as $\frac{1}{12}$, or $\frac{1}{2} =$ 6 one-twelfths or $6 \times \frac{1}{12}$

Compare twelfths with halves, fourths, sixths, etc. Have children insert symbols " < ", " > ", between each of the following pairs of fractional numbers:

$\frac{3}{12} \square \frac{1}{2}$; $\frac{3}{4} \square \frac{10}{12}$; $\frac{5}{6} \square \frac{5}{12}$; $\frac{3}{8} \square \frac{3}{12}$; etc.

6. Present a number ray. Ask children to locate the following points on the ray.

$\frac{3}{12}$, $\frac{5}{12}$, $\frac{7}{12}$, $\frac{9}{12}$; $1\frac{1}{12}$, $1\frac{5}{12}$, $2\frac{2}{12}$, $4\frac{7}{12}$, etc.

Have children find the distance between

$\frac{3}{12}$ and $\frac{9}{12}$; $\frac{4}{12}$ and $\frac{11}{12}$; $\frac{9}{12}$ and $1\frac{1}{12}$; $1\frac{4}{12}$ and $1\frac{5}{12}$

7. Fractions in Series; Using Number Line

Count forward and backward by one-twelfth; then change to equivalent fractions or mixed form.

Count forward and backward by one-twelfth, starting at any point on the line. Count forward and backward by groups of $\frac{2}{12}$, then $\frac{3}{12}$, then $\frac{4}{12}$, etc.

For example:

Counting forward - Groups of $\frac{2}{12}$

$\frac{1}{12}$, $\frac{3}{12}$, $\frac{5}{12}$, $\frac{7}{12}$, $\frac{9}{12}$, $\frac{11}{12}$ . . . . then $\frac{1}{12}$, $\frac{1}{4}$, $\frac{5}{12}$, $\frac{7}{12}$, $\frac{3}{4}$, $\frac{11}{12}$ . . .

$\frac{2}{12}$, $\frac{4}{12}$, $\frac{6}{12}$, $\frac{8}{12}$, $\frac{10}{12}$, $\frac{12}{12}$ . . . . then $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, 1, $1\frac{1}{6}$

Counting backward - Groups of $\frac{3}{12}$

$\frac{14}{12}$, $\frac{11}{12}$, $\frac{8}{12}$, $\frac{5}{12}$, $\frac{2}{12}$, ... then $1\frac{1}{6}$, $\frac{11}{12}$, $\frac{2}{3}$, $\frac{5}{12}$, $\frac{1}{6}$

Ask children

When counting by twelfths:

$1\frac{1}{12}$ follows __________ and comes before ________

Find the number that is 2 less than $5\frac{7}{12}$.

What number is one more than $3\frac{11}{12}$?

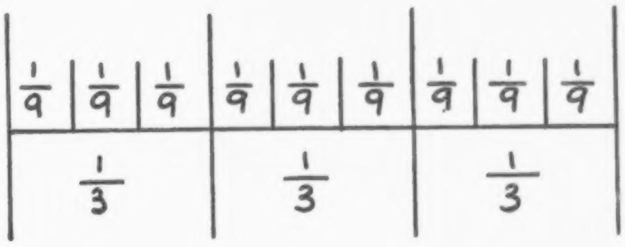
How many twelfths larger than $3\frac{1}{2}$ is $4\frac{1}{2}$?

How many twelfths must I add to $2\frac{1}{3}$ to reach the next whole number?

Ninths

1. Reinforce concept of thirds.

2. Children draw a line indicating a whole and discuss ways of dividing it into 9 equal parts. They discover that the line may be divided into thirds, then each third into 3 equal parts. A ninth is 1 third of 1 third.



Sevenths

Children draw a line indicating a whole, then divide it into seven equal parts.

Discuss reason why sixths, twelfths, etc., can be derived from halves, thirds, etc. but sevenths can not. Of what other fractional parts would this be true?

[eleventhe, thirteenths, etc.]

EVALUATION and / or PRACTICE
SUGGESTED EXERCISES

1. Complete each of the following.

$\frac{6}{12} = \frac{?}{2}$ $\quad \frac{6}{8} = \frac{?}{4}$ $\quad \frac{9}{12} = \frac{\square}{4}$ $\quad \frac{4}{6} = \frac{\square}{3}$

$\frac{4}{6} = \frac{?}{3}$ $\quad 1\frac{5}{9} = \frac{?}{9}$ $\quad 2\frac{5}{6} = \frac{n}{6}$ $\quad \frac{3}{9} = \frac{?}{3}$

2. Change the following to equivalent fractions with smaller denominators.

$\frac{8}{12} =$ ____ $\quad \frac{6}{8} =$ ____ $\quad \frac{6}{9} =$ ____

3. Change the following to equivalent fractions.

$\frac{1}{2} = \frac{?}{6}$ $\quad \frac{3}{4} = \frac{?}{8}$ $\quad \frac{2}{3} = \frac{?}{9}$ $\quad \frac{2}{3} = \frac{?}{12}$

4. Change these fractions to whole numbers or to mixed form.

$\frac{7}{7} = n$ $\quad \frac{8}{3} = n + \frac{\square}{3}$

$\frac{16}{8} = n$ $\quad \frac{16}{12} = n + \frac{\square}{3}$

5. Fill in the missing blanks in the following series:

$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ ____, $\quad \frac{1}{6}, \frac{1}{3}, \frac{1}{2},$ ____, $1\frac{5}{6}$.

$3, 2\frac{7}{8}, 2\frac{3}{4},$ ____, $2\frac{1}{2}$.

6. Answer the following questions:

$\frac{1}{4}$ is what part of $\frac{1}{8}$?

$\frac{1}{12}$ is what part of $\frac{1}{2}$?

$\frac{1}{2}$ is how many times as big as $\frac{1}{8}$?

$\frac{3}{4}$ is how many times as big as $\frac{1}{4}$?

$\frac{3}{4}$ is how many times as big as $\frac{1}{8}$?

7. Place a circle around the fraction on each line which has the greatest value:

$\frac{1}{2}$ . . . . . . $\frac{1}{4}$ . . . . . . $\frac{1}{8}$

$\frac{3}{4}$ . . . . . . $\frac{7}{8}$ . . . . . . $\frac{1}{2}$

$\frac{3}{8}$ . . . . . . $\frac{5}{6}$ . . . . . . $\frac{2}{3}$

$\frac{3}{4}$ . . . . . . $\frac{2}{3}$ . . . . . . $\frac{5}{6}$

8. Compare the following. Tell which is larger (smaller).
How much larger (smaller).

$\frac{1}{4}$ and $\frac{1}{8}$ $\frac{2}{3}$ and $\frac{5}{6}$

$\frac{3}{4}$ and $\frac{3}{8}$ $\frac{7}{8}$ and $\frac{3}{4}$

OPERATIONS

UNIT 34 - ADDITION OF FRACTIONAL NUMBERS: HORIZONTAL AND VERTICAL FORMAT

Objectives: To develop horizontal and vertical algorithms for addition of fractional numbers with unlike denominators.

To apply the Commutative and Associative Properties of Addition.

TEACHING SUGGESTIONS

1. Reinforce meaning of fractional numbers. For example:

$\frac{1}{2} + \frac{1}{2} = 1$ Is the sum a fractional number? Explain.

[Yes. Any whole number can be expressed as a fraction. $1 = \frac{1}{1}$ etc.]

2. Children should understand that the properties of Addition for the set of Whole Numbers are true for Addition for the set of Fractional Numbers.

Suggested exercises:

a. Find the sum of any two whole numbers. What kind of number is the sum? [Whole Number]

b. Find the sum of $\frac{1}{3} + \frac{1}{6}$. What kind of number is the sum? [Fractional Number]

c. Add: $\frac{1}{8} + \frac{0}{8} = n$

What do you notice about the sum? [It is the same as the first addend]

Explain. [$\frac{0}{8} = 0$ therefore adding $\frac{0}{8}$ is the same as adding 0]

d. Use number lines to show that: $\frac{3}{4} + \frac{1}{8} = \frac{1}{8} + \frac{3}{4}$

What property does this illustrate? [Commutative]

e. Show that:

$$\frac{1}{2} + \left(\frac{1}{2} + \frac{1}{4}\right) = \left(\frac{1}{2} + \frac{1}{2}\right) + \frac{1}{4}$$

What property does this illustrate? [Associative]

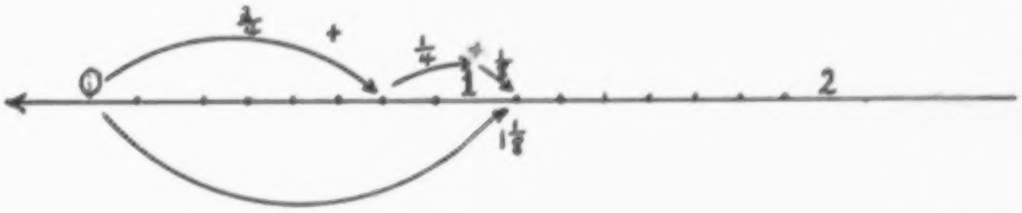
Explain.

Addition of Fractions With Unlike Denominators: Common Denominator Apparent; Horizontal Format

1. Introduce addition exercises through problem solving situations. Use number lines to help children in their thinking. For example:

Problem:

Mary needed 3 fourths of a yard of material for an apron and 3 eighths more for pockets and belt. How much cloth did she need? $\left(\frac{3}{4} + \frac{3}{8} = n\right)$



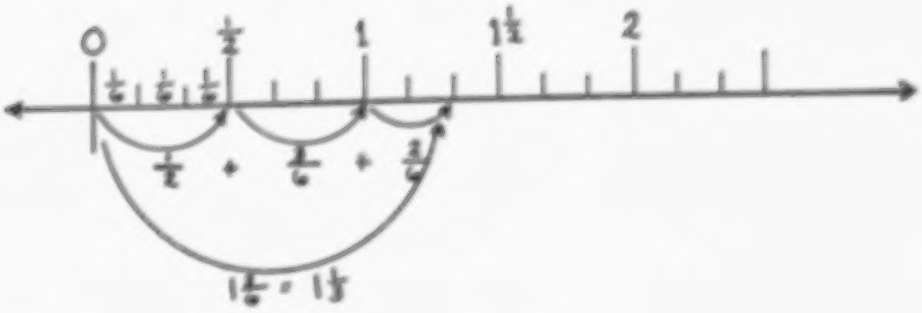
$$\frac{3}{4} + \frac{3}{8} \text{ as } \frac{3}{4} + \left(\frac{2}{8} + \frac{1}{8}\right) \text{; then}$$

$$\text{as} \left(\frac{3}{4} + \frac{1}{4}\right) + \frac{1}{8} \text{; as } 1,\ 1\frac{1}{8}$$

(Note use of Associative Property for Addition)

Have children find the solution for: $\frac{1}{2} + \frac{5}{6} = n$

a.


0
$\frac{1}{2}$
1
$1\frac{1}{2}$
2
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{2} + \frac{3}{6} + \frac{2}{6}$
$1\frac{2}{6} = 1\frac{1}{3}$

b. They should think through to complete the problem as follows:

$$\left(\frac{1}{2} + \frac{3}{6}\right) + \frac{2}{6} = 1\frac{2}{6} \qquad \text{or}$$

$$1\frac{2}{6} = 1\frac{1}{3}$$

or

$$\left(\frac{1}{2} + \frac{1}{2}\right) + \frac{1}{3} = 1\frac{1}{3}$$

Note the use of Associative Property for Addition.
$\frac{5}{6}$ was renamed as $\frac{3}{6} + \frac{2}{6}$.

2. Suggested type of practice exercises to maintain skill.
Use number line to check solutions.

Sixths and Wholes

$3\frac{1}{6} + 1 = n$ $\qquad$ $2\frac{3}{6} + 4 = n$

$4\frac{5}{6} + 2 = n$ $\qquad$ $3\frac{4}{6} + 2 = n$

Sixths and Halves

$\frac{2}{6} + \frac{1}{2} = n$

$\frac{1}{2} + \frac{4}{6} = n$

$2\frac{1}{2} + \frac{5}{6} = n$

$3\frac{1}{6} + 2\frac{1}{2} = n$

Sixths

$\frac{5}{6} + \frac{1}{6} = n$

$\frac{5}{6} + \frac{5}{6} = n$

$2\frac{1}{6} + \frac{5}{6} = n$

$4\frac{2}{6} + 2\frac{3}{6} = n$

Sixths and Thirds

$\frac{2}{6} + \frac{1}{3} = n$

$\frac{5}{6} + \frac{2}{3} = n$

$2\frac{3}{6} + \frac{1}{3} = n$

$3\frac{1}{6} + 2\frac{2}{3} = 5\frac{1}{6} + n = ?$

Encourage a variety of ways of evaluating.

For $5\frac{2}{3} + \frac{5}{6} = n$

$$\begin{aligned} &\left(5 + \frac{2}{3}\right) + \frac{5}{6} \\ &= 5 + \left(\frac{2}{3} + \frac{5}{6}\right) \\ &= 5 + \left(\frac{4}{6} + \frac{5}{6}\right) \\ &= 5\frac{9}{6} \\ &= 6\frac{1}{2} \end{aligned}$$

or

$$\begin{aligned} &5\frac{2}{3} + \left(\frac{2}{6} + \frac{3}{6}\right) \\ &= \left(5\frac{2}{3} + \frac{1}{3}\right) + \frac{1}{2} \\ &= 6\frac{1}{2} \end{aligned}$$

Vertical Format

1. Provide drill in renaming a whole number and a fraction. Include halves, fourths, eighths, thirds, sixths.

Have children complete the following:

$1\frac{5}{4} = 2\frac{\square}{4}$ $\qquad 9\frac{5}{2} = 11\frac{?}{2}$ $\qquad 13\frac{9}{8} = 14\frac{?}{8}$

$2\frac{5}{4} = 3\frac{\square}{4}$ $\qquad 10\frac{4}{2} = n$ $\qquad 18\frac{16}{8} = n$

$16\frac{5}{4} = 17\frac{\square}{4}$ $\qquad 13\frac{8}{2} = n$ $\qquad 10\frac{12}{8} = 11\frac{?}{8}$

$3 = 2\frac{\square}{3}$ $\qquad 2 = 1\frac{\square}{6}$

$3\frac{1}{3} = 2\frac{\square}{3}$ $\qquad 2\frac{1}{6} = 1\frac{\square}{6}$

$3\frac{2}{3} = 2\frac{\square}{3}$ $\qquad 2\frac{2}{6} = 1\frac{\square}{6}$

$1\frac{9}{8} = \square\frac{1}{8}$; $\qquad 3\frac{17}{8} = \square\frac{1}{8}$ $\qquad 7\frac{4}{3} = 8\frac{\square}{n}$; $\qquad 11\frac{8}{6} = 12\frac{\square}{n}$

2. Reinforce or introduce vertical algorithms for addition.

   Have children first estimate or arrive at the sum through "mental" computation.

   a. Fractions with Like Denominators

      Sums Less than 1

$$\begin{array}{c} \frac{2}{8} \\ \frac{1}{8} \\ \frac{2}{8} \\ \hline \frac{5}{8} \end{array}$$

      Sums Equal to or More than 1

$$\begin{array}{c} \frac{4}{8} \\ \frac{4}{8} \\ \hline \frac{8}{8} = 1 \end{array} \qquad \begin{array}{l} \frac{3}{8} \\ \frac{3}{8} \\ \frac{4}{8} \\ \hline \frac{10}{8} = 1\frac{2}{8} = 1\frac{1}{4} \end{array} \qquad \begin{array}{r} 8\frac{3}{8} \\ 7\frac{1}{8} \\ 15\frac{5}{8} \\ \hline 30\frac{9}{8} = 31\frac{1}{8} \end{array}$$

   b. Fractions with Unlike Denominators

      Use known equivalents

Sums Less than 1

$$\begin{array}{r}\frac{1}{3} = \frac{2}{6}\\ \frac{1}{6} = \frac{1}{6}\\ \hline \frac{3}{6} = \frac{1}{2}\end{array}$$

Sums Equal to or More than 1

$$\begin{array}{r}\frac{7}{8} = \frac{7}{8}\\ \frac{3}{4} = \frac{6}{8}\\ \hline \frac{13}{8} = 1\frac{5}{8}\end{array} \qquad \begin{array}{r}\frac{2}{3} = \frac{4}{6}\\ \frac{5}{6} = \frac{5}{6}\\ \frac{1}{2} = \frac{3}{6}\\ \hline \frac{12}{6} = 2\end{array}$$

c. Whole Number and Fraction

$$\begin{array}{r}27\\ 13\frac{2}{3}\\ \hline 40\frac{2}{3}\end{array} \qquad \begin{array}{r}\frac{3}{4}\\ 2\frac{1}{4}\\ \hline 2\frac{4}{4} = 3\end{array}$$

$$\begin{array}{r}\frac{4}{6} = \frac{4}{6}\\ 5\frac{2}{3} = 5\frac{4}{6}\\ \hline 5\frac{8}{6} = 6\frac{2}{6} = 6\frac{1}{3}\end{array} \qquad \begin{array}{r}7\frac{3}{4} = 7\frac{6}{8}\\ 2\frac{5}{8} = 2\frac{5}{8}\\ \hline 9\frac{11}{8} = 10\frac{3}{8}\end{array}$$

EVALUATION and / or PRACTICE
SUGGESTED EXERCISES

1. Find the sum for each pair of numbers and write its simplest fraction name. Use algorithms.

a. $\frac{3}{4}$, $\frac{1}{2}$ $\left[\frac{5}{4} = 1\frac{1}{4}\right]$ $11\frac{2}{3}$, $\frac{5}{6}$ $\left[12\frac{3}{6} = 12\frac{1}{2}\right]$

$\frac{3}{8}$, $5\frac{1}{4}$ $\left[5\frac{5}{8}\right]$

b.

$$\begin{array}{r} \frac{1}{2} \\ \frac{3}{8} \\ \frac{5}{8} \\ \hline \left[\frac{12}{8} = 1\frac{4}{8} = 1\frac{1}{2}\right] \end{array} \qquad \begin{array}{r} 4\frac{1}{2} \\ 7\frac{3}{4} \\ \hline \left[11\frac{5}{4} = 12\frac{1}{4}\right] \end{array} \qquad \begin{array}{r} 28\frac{2}{3} \\ 59\frac{5}{6} \\ \hline \left[88\frac{3}{6} = 88\frac{1}{2}\right] \end{array}$$

2. A magic square is one in which you can perform the operation on the numbers vertically, horizontally or diagonally and always get the same number for a result.

   Copy the square below.

   a. Add the numbers named by the fractions in each column and record the sum for each column. [10]

   b. Add the numbers named by the fractions in each row and record the sum for each row. [10]

   c. Begin in lower left hand corner. Add the numbers named by the fractions diagonally. Record their sum. [10]

   d. Begin in upper left hand corner. Add the numbers named by the fractions diagonally. Record their sum. [10]

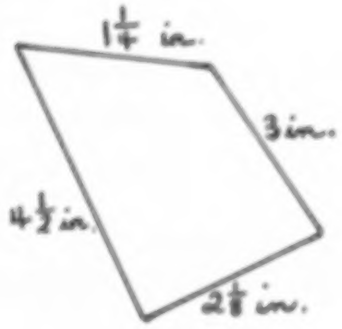
   e. Is each sum the same fractional number? What is the number? [10]

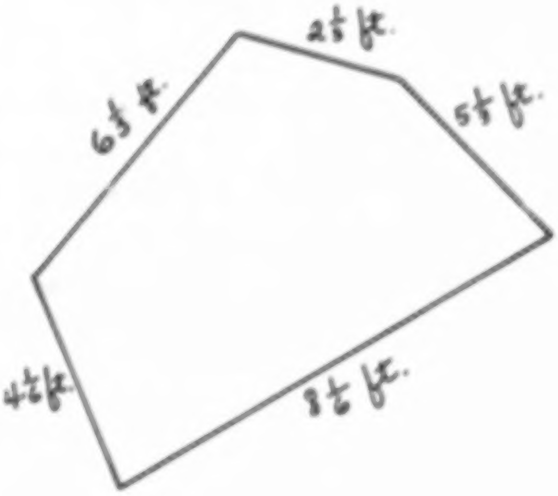
   f. Is the square a magic square? [yes]

$2\frac{1}{2}$	$3\frac{3}{8}$	$\frac{1}{2}$	$1\frac{3}{8}$	$2\frac{1}{4}$
$3\frac{1}{4}$	[illegible]	$1\frac{1}{4}$	$2\frac{1}{8}$	$2\frac{3}{8}$
$\frac{7}{8}$	$1\frac{1}{8}$	2	$2\frac{7}{8}$	$3\frac{1}{8}$
$1\frac{5}{8}$	$1\frac{7}{8}$	$2\frac{3}{4}$	3	$\frac{3}{4}$
$1\frac{3}{4}$	$2\frac{5}{8}$	$3\frac{1}{2}$	$\frac{5}{8}$	$1\frac{1}{2}$

3. John traveled $3\frac{1}{2}$ hours by plane and then $\frac{3}{4}$ of an hour by car to get to his grandmother's house. How many hours did he travel?

4. Find the perimeter of the following polygons.





[$10\frac{7}{8}$ in.]

[$26\frac{1}{3}$ ft.]

5. Write next to each step in the following algorithm either the property or operation involved. Indicate as well those steps in which renaming a fractional number occurred.

$$4\frac{1}{2} + 7\frac{3}{4} = n$$

a. $4\frac{1}{2} + 7\frac{3}{4} = \left(4 + \frac{1}{2}\right) + \left(7 + \frac{3}{4}\right)$

b. $= 4 + \left(\frac{1}{2} + 7\right) + \frac{3}{4}$ [associative property]

c. $= 4 + \left(7 + \frac{1}{2}\right) + \frac{3}{4}$ [Commutative property]

d. $= \left(4 + 7\right) + \left(\frac{1}{2} + \frac{3}{4}\right)$ [associative]

e. $= 11 + \left(\frac{2}{4} + \frac{3}{4}\right)$ [rename $\frac{1}{2}$; addition $(4 + 7)$]

f. $= 11 + \frac{5}{4}$ [addition]

g. $= 11 + \left(\frac{4}{4} + \frac{1}{4}\right)$ [renaming $\frac{5}{4}$]

h. $= 11 + \left(1 + \frac{1}{4}\right)$ $\left[\text{renaming } \frac{4}{4}\right]$

i. $= (11 + 1) + \frac{1}{4}$ [associative]

j. $= 12 + \frac{1}{4}$ [renaming 11 + 1]

k. $= 12\frac{1}{4}$ $\left[\text{renaming } 12 + \frac{1}{4}\right]$

6. Additional practice exercises may be found in textbooks and other printed material. Include problem situations that involve addition of fractional numbers.

OPERATIONS

UNIT 35 - SUBTRACTION OF FRACTIONAL NUMBERS: HORIZONTAL AND VERTICAL FORMAT

Objectives: To develop horizontal and vertical algorithms for subtraction of fractional numbers with unlike denominators.

To reinforce renaming numbers needed for subtraction with exchange.

To emphasize checking subtraction by adding.

TEACHING SUGGESTIONS

Halves, Fourths, Eighths, Thirds, Sixths

Horizontal Format - Subtraction of Fractional Numbers

1. Reinforce finding equivalent fractions for sixths, eighths, etc.

   For example: $\frac{6}{8} = \frac{3}{4}$; $\frac{7}{6} = 1\frac{1}{6}$; etc.

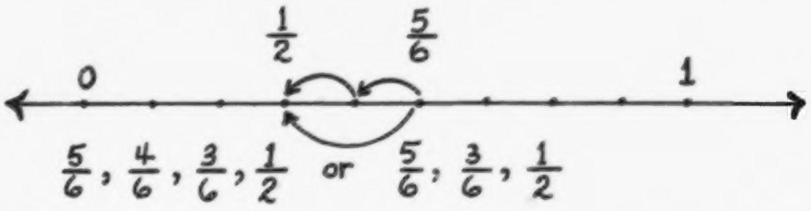
2. When children subtract "mentally" they generally begin with the entire minuend.

   Use number lines often.

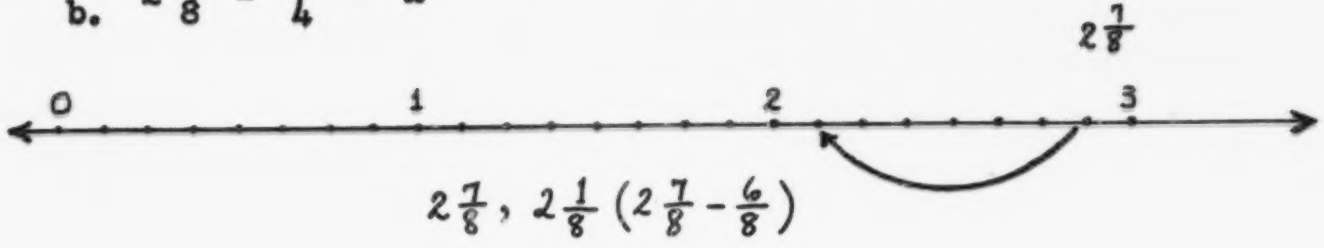
   Present a problem situation such as:

   a. Mary had $\frac{5}{6}$ of a yard of material. If she used $\frac{1}{3}$ of a yard to make a dress for her sister's doll, how much material did she have left?

$$\frac{5}{6} - \frac{1}{3} = n$$



b. $2\frac{7}{8} - \frac{3}{4} = n$



3. Have children solve the following open sentences and explain their solutions.

$$3\frac{5}{6} - 1\frac{1}{2} = 2\frac{5}{6} - n = ?$$

$$8\frac{5}{6} - 2\frac{2}{3} = 6\frac{5}{6} - \square = n$$

$$24\frac{3}{4} - 12\frac{1}{2} = 12\frac{6}{6} - \frac{4}{8} = n$$

Have children complete the following:

Sixths

$2\frac{1}{6} - 1 = \square$

$2\frac{1}{6} - \frac{1}{6} = \square$

$2\frac{1}{6} - 1\frac{1}{6} = \square$

$2\frac{1}{6} - \frac{3}{6} = \square$

$2\frac{1}{6} - 1\frac{3}{6} = \square$

Sixths and Halves

$3\frac{1}{6} - 1 = \square$

$3\frac{1}{6} - \frac{1}{2} = \square$

$3\frac{1}{6} - 1\frac{1}{2} = \square$

$3\frac{5}{6} - 1\frac{1}{2} = \square$

Sixths and Thirds

$\frac{5}{6} - \frac{1}{3} = n$

$\frac{9}{6} - \frac{2}{3} = n$

$3\frac{4}{6} - \frac{1}{3} = n$

$8\frac{5}{6} - 2\frac{2}{3} = n$

Verifying Solutions: Adding and Subtracting as Inverse Operations

Have children solve the additions below and check each by its related subtraction. For example:

$5\frac{2}{3}+\frac{5}{6}=n \quad \left[6\frac{1}{2}\right]$ $\qquad$ $2\frac{1}{8}+\frac{5}{8}=n \quad \left[2\frac{3}{4}\right]$

$6\frac{1}{2}-\frac{5}{6}=n \quad \left[5\frac{2}{3}\right]$ $\qquad$ $2\frac{3}{4}-\frac{5}{8}=n \quad \left[2\frac{1}{8}\right]$

Vertical Format

1. Reinforce or introduce vertical algorithm.

No Exchange

Minuends Less Than 1

$$\begin{array}{r} \frac{7}{8} \\ -\ \frac{3}{8} \\ \hline \frac{4}{8}=\frac{1}{2} \end{array} \qquad \begin{array}{rcr} \frac{3}{4} & = & \frac{6}{8} \\ -\frac{1}{8} & = & \frac{1}{8} \\ \cline{1-1}\cline{3-3} & & \frac{5}{8} \end{array}$$

Whole Number from Whole Number and Fraction

$$\begin{array}{r} 12\frac{5}{8} \\ -\ 9 \quad \\ \hline 3\frac{5}{8} \end{array} \qquad \begin{array}{r} 12\frac{4}{6} \\ -\ 7 \quad \\ \hline 5\frac{4}{6}=5\frac{2}{3} \end{array}$$

Fraction from Whole Number and Fraction

$$\begin{array}{r} 11\frac{4}{6} \\ -\quad \frac{3}{6} \\ \hline 11\frac{1}{6} \end{array} \qquad \begin{array}{rcr} 15\frac{7}{8} & = & 15\frac{7}{8} \\ -\ \frac{3}{4} & = & \frac{6}{8} \\ \cline{1-1}\cline{3-3} & & 15\frac{1}{8} \end{array}$$

Whole Number and Fraction from Whole Number and Fraction

$$\begin{array}{r} 18\frac{5}{6} \\ -\ 9\frac{2}{6} \\ \hline 9\frac{3}{6}=9\frac{1}{2} \end{array} \qquad \begin{array}{rcr} 13\frac{7}{8} & = & 13\frac{7}{8} \\ -\ 4\frac{3}{4} & = & 4\frac{6}{8} \\ \cline{1-1}\cline{3-3} & & 9\frac{1}{8} \end{array}$$

With Exchange - Related Denominators

1. Reinforce renaming.

   Have children complete the following:

$3 = \frac{\square}{6}$ $\qquad 3 = 2\frac{\square}{6}$ $\qquad 3\frac{1}{6} = 2\frac{\square}{6}$

$15\frac{5}{6} = 14\frac{\square}{6}$ $\qquad 22\frac{3}{4} = 21\frac{\square}{4}$ $\qquad 9\frac{1}{2} = 8\frac{\square}{4}$

2. As children solve problems they should discuss reasons for regrouping:

   a. Fraction from Whole number

$6 - \frac{1}{2} = n$ $\qquad 8 - \frac{3}{4} = n$

$$\begin{array}{rcr} 6 & = & 5\frac{2}{2} \\ -\frac{1}{2} & = & \frac{1}{2} \\ \hline & & 5\frac{1}{2} \end{array} \qquad \begin{array}{rcr} 8 & = & 7\frac{4}{4} \\ -\frac{3}{4} & = & \frac{3}{4} \\ \hline & & 7\frac{1}{4} \end{array}$$

Have children check their solutions.

   b. Whole Number and Fraction from Whole Number

$8 - 3\frac{5}{6} = n$ $\qquad 14 - 8\frac{2}{3} = n$

$$\begin{array}{rcr} 8 & = & 7\frac{6}{6} \\ -3\frac{5}{6} & = & 3\frac{5}{6} \\ \hline & & 4\frac{1}{6} \end{array} \qquad \begin{array}{rcr} 14 & = & 13\frac{3}{3} \\ -8\frac{2}{3} & & 8\frac{2}{3} \\ \hline & & 5\frac{1}{3} \end{array}$$

Have children check solutions.

c. **Whole Number and Fraction from Whole Number and Fraction**

Problem: There were $12\frac{1}{4}$ yds. of material in the stockroom. Our class used $6\frac{3}{4}$ yds. for aprons. How much material was left over?

Children may interpret the problem as a mathematical sentence.

$$\left[12\frac{1}{4} - 6\frac{3}{4} = n\right]$$

Estimate:

$n < 6$ and $n > 5$ n is between 5 and 6 $5 < n < 6$

Algorithm:

$$\begin{array}{rcrcl} 12\frac{1}{4} & = & 11\frac{5}{4} & & \\ \underline{-\;6\frac{3}{4}} & = & \underline{6\frac{3}{4}} & & \\ & & 5\frac{2}{4} & = & 5\frac{1}{2} \end{array}$$

d. Problems involving finding an equivalent fraction and then regrouping.

Problem: We had $3\frac{2}{3}$ yards of material. $\frac{5}{6}$ of a yard was cut off to make a tea apron. How much was left?

Children may interpret verbal problem as the open sentence:

$$3\frac{2}{3} - \frac{5}{6} = n$$

Estimate: $n > 2$

Algorithm:

$$\begin{array}{rcrcrl} 3\frac{2}{3} & = & 3\frac{4}{6} & = & 2\frac{10}{6} & \text{(Renaming and regrouping)} \\ \underline{-\;\frac{5}{6}} & = & \underline{-\;\frac{5}{6}} & = & \underline{-\;\frac{5}{6}} & \\ & & & & 2\frac{5}{6} & \end{array}$$

e. **Problem:** A baby weighed $7\frac{3}{4}$ pounds when he was born. At the end of the month he weighed $9\frac{1}{2}$ pounds. How much did he gain?

$$9\frac{1}{2} - 7\frac{3}{4} = n$$

Estimate: $n < 2$

Algorithm:

$$\begin{array}{rcrcr} 9\frac{1}{2} & = & 9\frac{2}{4} & = & 8\frac{6}{4} \\ \underline{-\,7\frac{3}{4}} & = & \underline{-7\frac{3}{4}} & = & \underline{-7\frac{3}{4}} \\ & & & & 1\frac{3}{4}\text{ lb.} \end{array}$$

Verifying Solutions

Children Check Subtracting by Adding

Since $15\frac{1}{2} - 7\frac{5}{6} = 7\frac{2}{3}$, $7\frac{2}{3} + 7\frac{5}{6}$ should equal $15\frac{1}{2}$

Continue to test and reinforce subtraction with fractions.

Further practice exercises and verbal problems may be found in textbooks and from situations found in other curriculum books, newspapers, etc.

OPERATIONS

UNIT 36 - MULTIPLICATION OF WHOLE NUMBERS: OBSERVING AND USING PATTERNS

Objectives: Using patterns to:
Develop understanding of the property of one in multiplication
Develop understanding of the property of zero in multiplication
Observe factor-factor-product relationships

TEACHING SUGGESTIONS

Property of "1" in Multiplication (Identity Element)

1. Have children complete each sentence below.

   a. Present open sentences with 1 as the first factor.

   1 x 1 = ☐
   1 x 2 = ☐
   1 x 3 = ☐
   1 x 4 = ☐

   1 x 24 = ☐
   1 x 189 = ☐
   1 x 1,000,000 = ☐
   1 x n = ☐ (n stands for any number)

   b. Present open sentences with 1 as the second factor.

   1 x 1 = ☐
   2 x 1 = ☐
   3 x 1 = ☐
   4 x 1 = ☐

   24 x 1 = ☐
   189 x 1 = ☐
   1,000,000 x 1 = ☐
   n x 1 = ☐ (n stands for any number)

2. Have children discuss the products. Try to get children to state the generalizations

   a. If one factor is "1", the product is the other factor.

   (1 x 6 = 6 or 6 x 1 = 6)
   (1 x a = a or a x 1 = a)

   b. "1" is a neutral (identity) element in multiplication.

Property of Zero in Multiplication

Zero as a factor.

1. Have children complete each of the following sentences and observe the pattern of products.

a. 1 x 3 = □   1 x 2 = □   1 x 1 = □   1 x 0 = □

1 x 0 = □
2 x 0 = □
3 x 0 = □
4 x 0 = □

16 x 0 = □
74 x 0 = □
123 x 0 = □
n x 0 = □ (n stands for any number)

b. 3 x 1 = □   2 x 1 = □   1 x 1 = □   0 x 1 = □

0 x 1 = □
0 x 2 = □
0 x 3 = □
0 x 4 = □

0 x 16 = □
0 x 74 = □
0 x 123 = □
0 x n = □ (n stands for any number)

Have children observe the product where zero is a factor.

2. Ask children to state the generalization:

If one factor is zero, the product is zero.

(a x 0 = 0 and 0 x a = 0)

Comparing Properties of Zero and One in Multiplication

1. Have children complete the sentences below

0 x 8 = n
1 x 8 = n

0 x 12 = n
1 x 12 = n

8 x 0 = n
8 x 1 = n

12 x 0 = n
12 x 1 = n

□ x 8 = 0
□ x 8 = 8

□ x 12 = 0
□ x 12 = 12

2. Ask children to verbalize the role of 0 and 1 in

a. multiplication

b. addition

Factor - Factor - Product Relationships

1. **Doubling the first factor; keeping the second factor constant**

2 x 12 = 24	3 x 12 = ☐
4 x 12 = 48	6 x 12 = ☐
8 x 12 = 96	12 x 12 = ☐
16 x 12 =192	24 x 12 = ☐

Ask children:

Which factor is kept constant?
Compare each first factor with the one preceding it. What pattern do you see?
If the second factor remains the same, and the first factor is doubled, what happens to the product?
What happens to the product of 2 numbers when the second factor is kept constant and the first factor is tripled? Multiplied by 4? by 5?

Write 4 other equations to show this relationship.

2. Doubling the second factor; keeping the first factor constant.

2 x 12 = 24	4 x 12 = ☐
2 x 24 = 48	4 x 24 = ☐
2 x 48 = 96	4 x 48 = ☐
2 x 96 = 192	4 x 96 = ☐

Children study the relationships among the first factors; the second factors.

If the first factor remains the same, and the second factor is doubled, what happens to the product?

What happens to the product when the first factor is kept constant and the second is tripled? Multiplied by 4? by 5?

Write 4 other equations to show this relationship.

Children state the generalization:

> When one factor is multiplied by a number, the product is multiplied by that same number.

3. Doubling both factors.

a. Present the following:

$$\begin{aligned} 2 \times 5 &= 10 \\ 4 \times 10 &= 40 \\ 8 \times 20 &= 160 \end{aligned}$$

Ask children to tell the relationship between:

The first factors of each equation
The second factors of each equation
The products of each equation

b. Have children compare, complete and compare again

$4 \times 10 = 40$ with $8 \times 20 = 160$
$8 \times 20 = 160$ with $16 \times 40 = 640$
$3 \times 5 = \square$ with $6 \times 10 = \square$
$6 \times 10 = \square$ with $12 \times 20 = \square$

Ask children to study the pattern and discuss

If both factors are doubled, what happens to the product? Tripled? Multiplied by 4? etc.

Ask children to state the generalization:

If both factors are multiplied by 2, the product is multiplied by 4.

If both factors are multiplied by 3, the product is multiplied by 9, etc.

Suggested Practice to Discover Patterns

Apply the following patterns to multiplying by sixes, sevens, eights and nines.

Ask children to explain patterns after each series.

a. **Doubling a Factor**

10 x 8 = □
20 x 8 = □ + □ = ?
40 x 8 = □ + □ = ?
80 x 8 = □ + □ = ?

b. **Adding 5 Eights**

5 x 8 = 40
10 x 8 = □ + □ = n
15 x 8 = □ + 40 = n
20 x 8 = □ + 40 = n

c. **Adding 10 Eights**

10 x 8 = □
20 x 8 = □ + 80 = n
30 x 8 = □ + 80 = n
40 x 8 = □ + 80 = n

d. **Doubling and then Adding 1, 2, 3, etc. Eights**

Since 6 x 8 = 48
Then 12 x 8 = n
And 13 x 8 = n

Since 10 x 8 = 80
Then 20 x 8 = □
And 22 x 8 = n

Since 20 x 8 = 160
Then 40 x 8 = □
And 43 x 8 = n

e. **Doubling**

20 x 70 = n
40 x 70 = □ + □ = n
80 x 70 = □ + □ = n

f. **Doubling and Adding Groups**

3 x 900 = n
6 x 900 = □ + □ = n
7 x 900 = □ + 900 = n

Money: Products Within $99.99

1. Be sure that children have acquired:

   Understanding of place value concepts of dollars and cents.
   Ability to regroup dollars, dimes and pennies.
   Ability to add by groups (mental computation).
   Ability to add with dollars and cents without and with exchange (written computation).

2. Drill additions that relate to multiplication.

3. Multiplying by 2

   2 x $ 2.30 = $ 2.30 + □ = ?
   2 x $ 4.31 = $ 4.31 + □ = ?
   2 x $12.62 = $12.62 + □ = ?

Doubling beginning with cents

$2 \times \$.65 = \$1.30$
$4 \times \$.65 = \square + \square = ?$
$8 \times \$.65 = \square + \square = ?$

Doubling - dollars and cents

$2 \times \$1.24 = \$2.48$
$4 \times \$1.24 = \square + \square = n$
$8 \times \$1.24 = \square + \square = n$
$6 \times \$4.13 = n$
$6 \times \$4.13 = n + n = \square$

OPERATIONS

UNIT 37 - MULTIPLICATION OF WHOLE NUMBERS: VERTICAL FORMAT; ONE FACTOR THROUGH NINE

NOTE TO TEACHER

Encourage children to devise and present their own methods of arriving at close estimates. Provide practice in estimating and / or arriving at exact products before computing.

Use a variety of directions:

Find the product of 321 and 8.
Multiply 321 by 8.
One factor is 321. The other factor is 8.
Find the product.

Objectives: To develop understanding of multiplication of whole numbers: one factor through 9; the other factor through 9999.

To help children understand that the vertical algorithm for multiplication makes use of the distributive property .

To help children develop skill in multiplying whole numbers involving dollars and cents.

TEACHING SUGGESTIONS

1. Evaluate and / or reinforce ability to multiply: one factor through 9, other factor through 999. For example:

$$\begin{array}{r} 68 \\ \underline{x\ 8} \end{array} \qquad \begin{array}{r} 728 \\ \underline{x\ 5} \end{array} \qquad \begin{array}{r} 486 \\ \underline{x\ 9} \end{array} \quad \text{etc.}$$

2. Extend understanding of multiplication: one factor through 9; other factor through 9999.

Sequence

No exchange $\begin{array}{r} 4214 \\ \underline{\times\ 2} \end{array}$

Exchange in Tens Place and / or Units Place

$$\begin{array}{rrrr} 1207 & 3274 & 2008 & 1040 \\ \underline{\times\ 4} & \underline{\times\ 3} & \underline{\times\ 4} & \underline{\times\ 7} \end{array}$$

Exchange of Hundreds for Thousands only

$$\begin{array}{rrr} 2632 & 1812 & 2934 \\ \underline{\times\ 3} & \underline{\times\ 4} & \underline{\times\ 2} \end{array}$$

Exchange in One, Two or Three Places

$$\begin{array}{rrrr} 1240 & 1432 & 2735 & 1254 \\ \underline{\times\ 5} & \underline{\times\ 4} & \underline{\times\ 3} & \underline{\times\ 6} \end{array}$$

Problem Situations:

a. A plane travels 2143 miles in one round trip. How many miles does it travel in 3 round trips?

$$3 \times 2143 = n$$

Estimate (Some children arrive at a closer estimate or the exact product.)

2143 is 2000 and 143
3 two thousands = 6 thousand
Estimate: $n > 6000$
or
2143 is over 21 hundred
3 times 21 hundred = 63 hundred
Estimate: $n > 6300$
or
n is between 6000 and 7000

Record the estimate.

Compute

$$\begin{array}{r} 2143 \\ \underline{\times\ 3} \\ 6429 \end{array}$$

Have children compare product with recorded estimate.

Children should verify solutions:

By Adding or by applying the Distributive Property

$$\begin{array}{r} 2143\\ 2143\\ \underline{2143}\\ 6429 \end{array} \qquad\qquad \begin{array}{r} 2143\\ \underline{\times\ 3}\\ 6429 \end{array} \qquad \begin{array}{rcr} 3 \times 2000 &=& 6000\\ 3 \times 100 &=& 300\\ 3 \times 40 &=& 120\\ 3 \times 3 &=& \underline{9}\\ && 6429 \end{array}$$

b. $\begin{array}{r} 2413\\ \underline{\times\ 3}\\ 39 \end{array}$ Record partial product for units, tens (39)
Discuss need for regrouping 12 hundred (3 x 400)
as 1 thousand 2 hundred

Present similar problems for discussion of regrouping hundreds for thousands.

Children continue to estimate, compute, check.

Gradually increase difficulties. Include more than one exchange.

3. Multiplication Exercises Involving Money.

Suggested Sequence:

No exchange . . . . .3 x $ 1.23
4 x $ 2.01
2 x $42.30 etc.

One Exchange. . . . . .3 x $ 2.26
4 x $12.82

Two Exchanges . . . . .7 x $ 1.38
5 x $14.60

Three Exchanges . . . . .2 x $ 6.58
4 x $23.64
7 x $12.75

a. Have children estimate before they compute. This helps them determine the correct placement of the decimal point in the product.

b. No Exchange

Problem: What is the cost of 2 baseballs at $2.34 each?

$2 \times \$2.34 = n$

Children record estimate.
$n > \$4$ (2 x $2) or as
$n > \$4.60$ (2 x $2.30)

Have children compare the addition form with the multiplication form.

$$\begin{array}{r} \$2.34 \\ \underline{2.34} \\ \$4.68 \end{array} \qquad\qquad \begin{array}{r} \$2.34 \\ \underline{\times\ 2} \\ \$4.68 \end{array}$$

As children multiply they should relate each step to the addition, beginning with the pennies.

Discuss the placement of the decimal point to separate the dollars from the cents.

Compare the product with the estimate.

c. Exchange (Dimes for dollars)

Follow the procedure suggested for "No Exchange"

$$\begin{array}{r} \$12.63 \\ 12.63 \\ \underline{12.63} \end{array} \qquad \begin{array}{r} \$12.63 \\ \underline{\times\ 3} \end{array}$$

Estimate: $n > \$36$ (3x$12)
$n > \$37.50$ (3x$12.50)
$n > \$37.80$ (3x$12.60)

As the product is recorded the teacher should direct the children's attention to the 18 dimes (3 x 6 dimes) which has been regrouped as 1 dollar and 8 dimes.

When understanding is assured, use the multiplication form mainly.

Have children check the product by:

Adding if the multiplier is 2, 3, 4, or 5.

$$\begin{array}{r} \$12.63 \\ 12.63 \\ \underline{12.63} \\ \$37.89 \end{array}$$

Applying the Distributive Property

$$\begin{array}{rcr} 3 \times \$12 & = & \$36.00 \\ \underline{3 \times \$\ .63} & \underline{=} & \underline{1.89} \\ & & \$37.89 \end{array}$$

d. Extend to multiplications involving 2 or 3 exchanges.

4. Additional problems will be found in textbooks.

OPERATIONS

UNIT 38 - MULTIPLICATION OF WHOLE NUMBERS: DEVELOPING GENERALIZATIONS

Objective: To help children formulate generalizations for multiplying with 10; 100; 1000 and their multiples.

TEACHING SUGGESTIONS

Multiplying with 10 as One Factor

1. Present the following, recording products as children state them.

$$\begin{array}{cccccc} \multicolumn{2}{c}{A} & \multicolumn{2}{c}{B} & \multicolumn{2}{c}{C} \\ \begin{array}{r} 3 \\ \times\ 10 \\ \hline \end{array} & \begin{array}{r} 10 \\ \times\ 3 \\ \hline \end{array} & \begin{array}{r} 5 \\ \times\ 10 \\ \hline \end{array} & \begin{array}{r} 10 \\ \times\ 5 \\ \hline \end{array} & \begin{array}{r} 8 \\ \times\ 10 \\ \hline \end{array} & \begin{array}{r} 10 \\ \times\ 8 \\ \hline \end{array} \end{array}$$

Question children about the product when 10 is one factor. They should note:

Similarity of products in each pair of expressions.
Position of zero in product when 10 is a factor.
Position of digit representing the other factor.

2. Repeat procedure above: One factor 10, the other a two digit number.

$$\begin{array}{cccccc} \multicolumn{2}{c}{A} & \multicolumn{2}{c}{B} & \multicolumn{2}{c}{C} \\ \begin{array}{r} 11 \\ \times 10 \\ \hline \end{array} & \begin{array}{r} 10 \\ \times 11 \\ \hline \end{array} & \begin{array}{r} 14 \\ \times 10 \\ \hline \end{array} & \begin{array}{r} 10 \\ \times 14 \\ \hline \end{array} & \begin{array}{r} 32 \\ \times 10 \\ \hline \end{array} & \begin{array}{r} 10 \\ \times 32 \\ \hline \end{array} \end{array}$$

Have children record products and observe:

The zero in the products always occurs in ones place.
The numeral in the other factor appears one place to the left in the product.
When one factor is 10 the product is ten times greater than the other factor.

3. Record a series of examples, such as:

16	10	27	10	28	
x10	x19	x10	x35	x10	etc.

Ask children whether they can arrive at the products in a faster or shorter way.

Record the products as the children express their thinking.

The children observe that when 10 is a factor the same pattern for the product emerges.

4. Have children express the generalizations in their own words.

5. Provide for practice using both the vertical and horizontal forms.

Multiplying with Multiples of 10 as One Factor

1. Reinforce and / or teach finding sums needed to derive unknown products from known products by applying the Distributive Property of Multiplication with respect to Addition.

   In order to find a solution using the Distributive Property, children should have the ability to find required sums quickly. For example, To find the product for

   20 x 13 = n   children should know that:
   10 x 13 = 130   and that another
   10 x 13 = 130   Then
   130 + 130 will equal 260 or, 20 x 13

   Suggested exercises to reinforce finding sums. Multiples of thirteen are used here. Adapt to other numbers as required.

   a. Adding 10 Thirteens to multiples of 10 Thirteens.

   130 + 130 = ?
   260 + 130 = ? . . . (20 x13) + (10 x 13) or 30 x 13
   390 + 130 = ?
   520 + 130 = ?
   650 + 130 = ?
   through
   1040 + 130 = ?

b. Doubling multiples of 10 thirteens

130 + 130 = ?
520 + 520 = ? (40 x 13) + (40 x 13) or 80 x 13
390 + 390 = ?
260 + 260 = ?

2. As children solve the following exercises the teacher should record some of their thinking. For example,

30 x 13 as:

20 x 13 = 260
30 x 13 = 260 + 130 = 390

or as

$$\begin{array}{r} 20 \times 13 = 260 \\ \underline{10 \times 13 = 130} \\ 30 \times 13 = 390 \end{array}$$

80 x 13 as:

40 x 13 = 520
80 x 13 = 520 + 520 = 1040

or as

$$\begin{array}{r} 40 \times 13 = 520 \\ \underline{40 \times 13 = 520} \\ 80 \times 13 = 1040 \end{array}$$

a.

A

10 x 13 = □
20 x 13 = □ + 130 = n

through

90 x 13 = □ + 130 = n

B

10 x 13 = □
20 x 13 = □ + □ = n
40 x 13 = □ + □ = n
80 x 13 = □ + □ = n

C

20 x 13 = □ + 130 = n
40 x 13 = □ + 260 = n
60 x 13 = □ + 260 = n
80 x 13 = □ + 260 = n

D

20 x 13 = □ + □ = n
40 x 13 = □ + □ = n
50 x 13 = □ + 130 = n

E

20 x 13 = □ + □ = n
40 x 13 = □ + □ = n
80 x 13 = □ + □ = n
90 x 13 = □ + □ = n

b. Since 10 x 13 = 130
5 x 13 = ?

Since 10 x 13 = 130
15 x 13 = ?
etc.

Since 10 x 13 = 130
20 x 13 = n

Since 20 x 13 = 260
21 x 13 = n

Since 10 x 13 = 130
20 x 13 = ?
30 x 13 = ?
40 x 13 = ?
42 x 13 = ?

Since 100 x 13 = 1300
Then 50 x 13 = ?

Since 10 x 13 = 130
11 x 13 = ?

Since 10 x 13 = 130
13 x 13 = ?
etc.

Since 20 x 13 = 260
40 x 13 = ?

Since 30 x 13 = 390
60 x 13 = ?

Since 10 x 13 = 130
20 x 13 = ?
40 x 13 = ?
80 x 13 = ?
81 x 13 = ?

3. Develop generalization for multiplying with multiples of 10 as one factor; 20, 30, 40, 50 . . . 90

Present the following exercises recording products as children state them.

20	30	40	50	
x 5	x 5	x 5	x 5	etc.

Have children observe that:

The zero in the product always occurs in ones place.

The digits in tens and hundreds places are arrived at by multiplying the number of tens and the other factor.

Present the following exercises.

20	20	20	20	20	20		20
x10	x20	x30	x40	x50	x60	through	x90

Children record the products.

They should observe that the zero in the product always occurs in ones place.

They compare the product with the factors in $\begin{array}{r} 20 \\ \underline{\times 10} \end{array}$

and in $\begin{array}{r} 20 \\ \underline{\times 20} \\ 400 \end{array}$

They should see that when one of the factors is a multiple of 10, the digits in the other places are arrived at by multiplying the number of tens.

For example:

	Step 1 Record Zero	Step 2 Multiply by Number of Tens
$\begin{array}{r} 20 \\ \underline{\times 30} \end{array}$	$\begin{array}{r} 20 \\ \underline{\times 30} \\ 0 \end{array}$	$\begin{array}{r} 20 \\ \underline{\times 30} \\ 600 \end{array}$
$\begin{array}{r} 20 \\ \underline{\times 90} \end{array}$	$\begin{array}{r} 20 \\ \underline{\times 90} \\ 0 \end{array}$	$\begin{array}{r} 20 \\ \underline{\times 90} \\ 1800 \end{array}$

Encourage children to formulate the generalization in their own words.

For example, When multiplying by 10 or multiples of ten record the zero in the ones place and multiply by the number of tens.

One Factor 100

1. Present orally

	Record only	
Since 50 twos = 100 Then 100 twos = ?	$\begin{array}{r} 2 \\ \underline{\times 100} \end{array}$	$\begin{array}{r} 100 \\ \underline{\times 2} \end{array}$
Since 50 sevens = 350 Then 100 sevens = ?	$\begin{array}{r} 7 \\ \underline{\times 100} \end{array}$	$\begin{array}{r} 100 \\ \underline{\times 7} \end{array}$
Since 50 fours = 200 Then 100 fours = ?	$\begin{array}{r} 4 \\ \underline{\times 100} \end{array}$	$\begin{array}{r} 100 \\ \underline{\times 4} \end{array}$

Question children about the product when 100 is a factor.
(See development used when 10 is a factor)

Children should compare the numerals representing the factors with the position of the numerals in the products.

Encourage children to derive the generalization in their own words. For example,

> When we multiply, and one factor is 100 the numerals in the other factor appear again in the product two places to the left. There are two zeros - one in units place and the other in Tens place.

2. Provide practice. Use both vertical and horizontal forms.

100 x 4 = n
100 x 9 = n

5 x 100 = n
8 x 100 = n

$$\begin{array}{r} 4 \\ \underline{\times 100} \end{array} \qquad \begin{array}{r} 100 \\ \underline{\times\ 8} \end{array} \qquad \begin{array}{r} 6 \\ \underline{\times 100} \end{array} \text{ etc.}$$

One Factor: Multiple of 100

1. Children arrive at unknown products by adding known products.

Adding in Sequence

100 x 4 = n
200 x 4 = n + 400 = □
300 x 4 = n + 400 = □
400 x 4 = n + 400 = □
through
900 x 4 = n + 400 = □

Other Patterns

200 x 4 = n
400 x 4 = n + n = □
600 x 4 = n + 800 = □
800 x 4 = n + 800 = □

2. Record as follows:

$$\begin{array}{r} 4 \\ \underline{\times 100} \end{array} \qquad \begin{array}{r} 4 \\ \underline{\times 200} \end{array} \qquad \begin{array}{r} 4 \\ \underline{\times 300} \end{array} \qquad \begin{array}{r} 4 \\ \underline{\times 400} \end{array}$$

$$\begin{array}{r} 100 \\ \underline{\times\ \ 4} \end{array} \qquad \begin{array}{r} 200 \\ \underline{\times\ \ 4} \end{array} \qquad \begin{array}{r} 300 \\ \underline{\times\ \ 4} \end{array} \qquad \begin{array}{r} 400 \\ \underline{\times\ \ 4} \end{array}$$

Children observe and discuss products.
They express the generalization for multiplying with multiples of 100 in their own words.

For example: When multiplying by 100 or multiples of 100, record the zeros in the ones and tens places, then multiply by the number of hundreds. (2, 3, 4, etc.)

OPERATIONS

$$\begin{tikzcd} + \arrow[r] \arrow[d, leftrightarrow] & \times \arrow[d, leftrightarrow] \\ - \arrow[r] & \div \end{tikzcd}$$

UNIT 39 - DIVISION OF WHOLE NUMBERS

Objectives: To extend division of whole numbers: Vertical algorithm; Quotients through 99; Then through 999; Divisors through 9.

TEACHING SUGGESTIONS

1. Drill multiplication and division facts in specific patterns.

   The following are suggestions for various types of drill.

   a. Find the missing factor.

   Since $5 \times 7 = 35$
   $\square \times 7 = 42$

   Since $15 \times 7 = 105$
   $\square \times 7 = 112$

   Since $25 \times 7 = 175$
   $\square \times 7 = 182$

   Since $9 \times 7 = 63$
   $\square \times 7 = 70$

   Since $100 \times 7 = 700$
   $\square \times 7 = 707$

   Since $3000 \times 7 = 21000$
   $\begin{array}{r}\square\\ 7\overline{)21007}\end{array}$

   b. Using the Distributive Property of Division with respect to Addition to arrive at unknown quotients from known quotients.

   The following exercises are for dividing by eights.
   Adapt for other facts.

   Supply correct quotients.

   $8\overline{)192}$ as $8\overline{)160 + 32}$ $\quad$ $8\overline{)192}$ as $8\overline{)96 + 96}$
   $8\overline{)2000}$ as $8\overline{)1600 + 400}$ $\quad$ $8\overline{)2000}$ as $8\overline{)1000 + 1000}$

   Have children complete the following exercises:

   $128 \div 4 = n$
   as: $(120 + 8) \div 4 = n$
   $(\square \div 4) + (\Delta \div 4) = n$

   $637 \div 7 = n$
   as: $(630 + 7) \div 7 = n$
   $(\square \div 7) + (\Delta \div 7) = n$
   $\square + \Delta = n$

c. Using the Distributive Property of Division with respect to Subtraction.

Have children complete the following:

$57 \div 3 = n$ $\qquad$ $3\overline{)87}$ as $3\overline{)90} - 3\overline{)3}$
as: $(60 - 3) \div 3 = n$
$(60 \div 3) - (3 \div 3) = n$ $\qquad$ $3\overline{)117}$ as $3\overline{)120} - 3\overline{)3}$
$\square - \Delta = n$

d. Doubling and adding eights. Adapt for other facts.

Necessary background: Adding eight and multiples of eight.

Drill Exercises

$8 + 8 = n$ $\quad$ $16 + 16 = n$ $\quad$ $32 + 32 = n$ $\quad$ $64 + 64 = n$ etc.
$24 + 24 = n$ $\quad$ $48 + 48 = n$ $\quad$ $96 + 96 = n$ $\quad$ $192 + 192 = n$ etc.

Complete the following:

$4 \times 8 = n$ $\quad$ $10 \times 8 = n$ $\quad$ $40 \times 8 = n$
$8 \times 8 = n$ $\quad$ $20 \times 8 = n$ $\quad$ $80 \times 8 = n$
$16 \times 8 = n$ $\quad$ $25 \times 8 = n$ $\quad$ $89 \times 8 = n$
$32 \times 8 = n$ $\quad$ $25\frac{1}{2} \times 8 = n$

$\square \times 8 = 16$ or $8\overline{)16}$ $\quad$ $\square \times 8 = 24$ or $8\overline{)24}$
$\square \times 8 = 32$ or $8\overline{)32}$ $\quad$ $\square \times 8 = 48$ or $8\overline{)48}$
$\square \times 8 = 64$ or $8\overline{)64}$ $\quad$ $\square \times 8 = 56$ or $8\overline{)56}$

e. Subtracting eights from multiples of eight.

When multiplication and division facts are drilled through subtraction, children must know the subtraction facts and extensions involved.

Necessary background: $80 - 8$, $\quad 160 - 8$, $\quad 400 - 8$,
$800 - 8$, $\quad 800 - 80$

Complete the following:

$10 \times 8 = n$ $\quad$ $20 \times 8 = n$ $\quad$ $50 \times 8 = n$ $\quad$ $100 \times 8 = n$
$9 \times 8 = n$ $\quad$ $19 \times 8 = n$ $\quad$ $49 \times 8 = n$ $\quad$ $90 \times 8 = n$

✱ f. Dividing one factor by 2

$10 \times 8 = n$ $\quad 50 \times 8 = n$ $\quad 100 \times 14 = n$ [1400]
$5 \times 8 = n$ $\quad 25 \times 8 = n$ $\quad 50 \times 14 = n$ [700]
$2\frac{1}{2} \times 8 = n$ $\quad 12\frac{1}{2} \times 8 = n$ $\quad 25 \times 14 = n$ [350]
$1\frac{1}{4} \times 8 = n$ $\quad 6\frac{1}{4} \times 8 = n$ $\quad 12\frac{1}{2} \times 14 = n$ [175]

Have children observe, then state generalization:

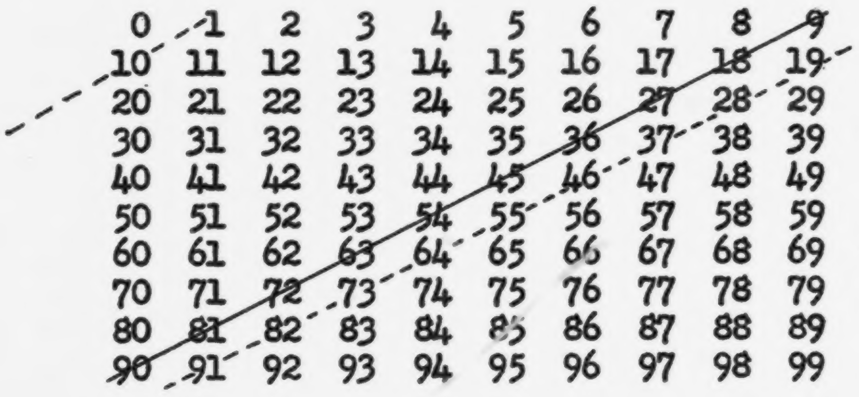
"When one factor is divided by 2 and the other factor is not changed, what happens to the product?"

2. Drills for Division Without and With Remainders

a. Using a number chart

Present a chart showing numbers from 0 to 99. Duplicate a quantity of these charts for later use with other "tables".

Have children draw a line through all the multiples of 9.
(9, 18, 27, 36, etc)



0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Children draw line segments, colored differently, to show which numbers have a remainder of 1 when divided by 9.

Children draw other lines, colored differently, to show a remainder of 2, of 3, etc. They note the patterns that emerge.

Present other number charts similar to the one above, one at a time. Follow the same procedure for multiples of other numbers.

Children name the set of numbers that have a remainder of 0, when divided by 9.
[0, 9, 18, 27, 36, . . .]

Children name the set of numbers that have a remainder of 1, when divided by nine. They note the relationship to 9 and multiples of 9.

[1, 10, 19, 28, 37, . . .]

Children name the set of numbers that have a remainder of 2, when divided by nine. They note the relationship to 9 and multiples of 9.

[2, 11, 20, 29, 38, . . .]

Follow the same procedure for dividing by other numbers.

b. Reinforce properties that apply to division

Have children mark the following true or false.
If false make it true.

$$\square \div \Delta = \Delta \div \square$$

$$64 \div 1 = 64$$

$$84 \div 3 = (60 \div 3) + (24 \div 3)$$

c. If a dividend is a multiple of the divisor, is there a remainder? When does a division exercise have a remainder?

3. Algorithms: Divisors through 9; Quotients through 99

Gradually increase the difficulty by including more difficult quotients. For example:

$$3\overline{)257} \qquad 6\overline{)452} \qquad 9\overline{)834}$$

Encourage children to shorten the recording by using larger, therefore, fewer multiples of ten.

From:

$$\begin{array}{r|l} 75 & \\ \hline 6\overline{)452} & \\ \underline{120} & 20 \\ 332 & \\ \underline{240} & 40 \\ 92 & \\ \underline{60} & 10 \\ 32 & \\ \underline{30} & \underline{5} \\ 2 & 75 \end{array}$$

Encourage

$$\begin{array}{r|l} 75 & \\ \hline 6\overline{)452} & \\ \underline{300} & 50 \\ 152 & \\ \underline{120} & 20 \\ 32 & \\ \underline{30} & \underline{5} \\ 2 & 75 \end{array}$$

Advance to

$$\begin{array}{r|l} 75 & \\ \hline 6\overline{)452} & \\ \underline{420} & 70 \\ 32 & \\ \underline{30} & \underline{5} \\ 2 & 75 \end{array}$$

Solution: 75 R 2

4. Algorithms: Quotients through 999; Divisors through 9

a. Skill in multiplying by 100 and multiples of 100 is basic to the development of division situations which involve 3-place quotients. To help children acquire this skill, the following development is suggested:
Drill: Using Relationships

Present one pattern at a time
Have children record products only
Record children's method for arriving at products

Multiplying in Sequence

$100 \times 4 = n$
$200 \times 4 = n$
$300 \times 4 = n$
$400 \times 4 = n$
through
$900 \times 4 = n$

Doubling

$100 \times 4 = n$
$200 \times 4 = n$
$400 \times 4 = n$
$800 \times 4 =$(1600 + 1600 read as 16 hundred + 16 hundred)

Other Sequences

$100 \times 4 = n$
$200 \times 4 = n$
$400 \times 4 = n$
$600 \times 4 = n$
$800 \times 4 = n$

$100 \times 4 = n$
$300 \times 4 = n$
$900 \times 4 =$ (12 hundred three times)

$100 \times 4 = n$
$500 \times 4 = n$
$600 \times 4 = n$
$700 \times 4 = n$
etc.

b. Suggested Development

Present a problem situation (quotient between 100 and 200) such as:

The P.T.A. contributed 986 cupcakes for the school Cake Sale. They are to be packed in cellophane bags with 8 cupcakes in each bag. How many bags will the school have to sell?

Ask the children to:

State the action involved
State the process to be used
Record the symbols to be used to solve the problem
[$\square \times 8 = 986$; $8\overline{)986}$; $986 \div 8 = n$]
Tell what the symbols say. [How many eights in 986?]

Ask some children how many eights to begin with
Discuss with children and record several ways of solving

Various methods of solving follow:

$$\begin{array}{r l} 123 & \\ 8\overline{)986} & \\ \underline{160} & 20 \\ 826 & \\ \underline{320} & 40 \\ 506 & \\ \underline{320} & 40 \\ 186 & \\ \underline{160} & 20 \\ 26 & \\ \underline{24} & \underline{3} \\ 2 & 123 \end{array} \qquad \begin{array}{r|l} 123 & \\ 8\overline{)986} & \\ \underline{320} & 40 \\ 666 & \\ \underline{640} & 80 \\ 26 & \\ \underline{24} & \underline{3} \\ 2 & 123 \end{array} \qquad \begin{array}{r|l} 123 & \\ 8\overline{)986} & \\ \underline{400} & 50 \\ 586 & \\ \underline{400} & 50 \\ 186 & \\ \underline{160} & 20 \\ 26 & \\ \underline{24} & \underline{3} \\ 2 & 123 \end{array}$$

Solution: 123 bags and 2 extra cupcakes.

Children will continue to arrive at solutions in a variety of ways depending upon their level of ability.

Using 100 as the first partial quotient.

Discuss the various solutions above.
Encourage children to begin with a still greater number of eights.
When 100 eights are suggested, present the following:

$$\begin{array}{r|l} 123 & \\ 8\overline{)986} & \\ \underline{800} & 100 \\ 186 & \\ \underline{160} & 20 \\ 26 & \\ \underline{24} & \underline{3} \\ 2 & 123 \end{array}$$

Compare with solutions above.
Discuss the shortened form.

Check by using the Distributive Property for Multiplication with respect to addition.

$$\begin{array}{r} 100 \times 8 = 800 \\ 20 \times 8 = 160 \\ 3 \times 8 = \underline{24} \\ 984 \end{array} \qquad \begin{array}{r l} 984 & \\ \underline{+\ 2} & \text{(extra cupcakes)} \\ 986 & \end{array}$$

Provide for practice. Select divisions in which 100 is the first partial quotient, e.g.,

$7\overline{)864}$ $3\overline{)528}$ $6\overline{)852}$ $4\overline{)611}$ etc.

5. Estimating: Estimating should precede all computation

Provide practice in estimating the size of the quotient when the dividend is in the hundreds.

Will the quotient be in the tens or in the hundreds?
For example:

$7\overline{)317}$ and $7\overline{)864}$

Children decide upon the number of groups with which to begin, by comparing the size of the dividend with the number of groups selected. Children think as follows:

For $7\overline{)317}$ Will there be as many as 10 sevens?
(10 x 7 = 70) too small
Will there be as many as 100 sevens?
(100 x 7 = 700) too large

Continue to compute using few or many multiples of 10 depending upon their ability.

For $7\overline{)864}$ Will there be as many as 10 sevens?
(10 x 7 = 70) too small
Will there be as many as 100 sevens?
(100 x 7 = 700) Yes.

Continue the computation beginning with 100 as the first recorded quotient.

6. More Difficult Computations

a. Present a problem such as:

Children contributed 716 used books for the White Elephant Sale held by the P.T.A. Their mothers tied them into bundles of 3 each. How many bundles did they tie?

Children may use hundreds repeatedly in the partial quotients. Begin with 100 threes. 416 is recorded as a partial dividend. Then ask what large group of threes might be removed next?

```
      238
   3)716 |
     300 | 100
     416 |
     300 | 100
     116 |
      90 |  30
      26 |
      24 |   8
       2 | 238
```

Complete the division.

Solution: 238 bundles and 2 extra books

b. Present similar division situations

Discuss the correct placement of the quotients after completing the computations. See examples below.

```
     234
  4)936 |
    400 | 100
    536 |
    400 | 100
    136 |
    120 |  30
     16 |
     16 |   4
      0 | 234
```

```
      242
   6)1457 |
      600 | 100
      857 |
      600 | 100
      257 |
      240 |  40
       17 |
       12 |   2
        5 | 242
```

```
      323
   5)1619 |
      500 | 100
     1119 |
      500 | 100
      619 |
      500 | 100
      119 |
      100 |  20
       19 |
       15 |   3
        4 | 323
```

d. Children who have acquired skill in multiplying by multiples of 100 and who understand the meaning of the numbers they will use in division, may be ready to shorten the recording.

For example, children think as follows:

```
      242
   6)1457 |
     1200 | 200
      257 |
      240 |  40
       17 |
       12 |   2
        5 | 242
```

Will there be as many as 10 sixes?
    Yes, but 60 is too small.
Will there be as many as 100 sixes?
    (100 x 6 = 600) Yes
Will there be as many as 200 sixes?
    (200 x 6 = 1200) Yes
Will there be as many as 300 sixes?
    (300 x 6 = 1800) No, too large

Estimate: There will be over 200 groups. Children record 200 and continue the computation.

EVALUATION and / or PRACTICE
SUGGESTED EXERCISES

1. Find quotients

$6\overline{)805}$ $7\overline{)869}$ $3\overline{)642}$

$4\overline{)1373}$ $5\overline{)2228}$

$6\overline{)3927}$ $7\overline{)3269}$ $3\overline{)1718}$

$4\overline{)3878}$ $9\overline{)5149}$

2. Additional exercises and problems may be found in text books.

SETS; NUMBER; NUMERATION

UNIT 40 - CONCEPTS OF EQUIVALENT FRACTIONS EXTENDED; APPLYING THE MULTIPLICATIVE PROPERTY OF "1".

NOTE TO TEACHER

"Equal fractions", by definition have equal numerators and equal denominators.

e.g. $\frac{1}{3} = \frac{1}{3}$ $\quad \frac{3}{6} = \frac{2+1}{4+2}$

"Equivalent fractions" are fractions (numerals) that name the same fractional number.

e.g. $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$

In this document "equal fractions" are rarely considered.

The Identity Property for Multiplication states that if a number is multiplied by 1 (The Identity Element for Multiplication), the value of the number is not changed. The Identity Property applies to fractional numbers as well as whole numbers.

$$4 \times 1 = 4; \quad \frac{2}{3} \times 1 = \frac{2}{3}$$

1 may be renamed as a fraction, thus:

$$\frac{2}{3} = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6} \qquad \frac{a}{b} = \frac{a}{b} \times \frac{c}{c} = \frac{ac}{bc} \quad b \neq 0, c \neq 0$$

This illustrates the Fundamental Principle of Fractions:

MULTIPLYING NUMERATOR AND DENOMINATOR OF A FRACTION BY THE SAME NUMBER, NOT ZERO, DOES NOT CHANGE THE VALUE OF THE FRACTION.

__Objective:__ To help children understand that changing a fraction to higher terms or to its simplest form involves the Multiplicative Property of "1".

TEACHING SUGGESTIONS

Renaming Fractions: Higher Terms

1. Have children rename $\frac{1}{6}$ as twelfths. $\left[\frac{1}{6} = \frac{2}{12}\right]$

   Ask children:

   The denominator 12 is how many times as large as the denominator 6? [twice as large]

   The numerator 2 is how many times as large as the numerator 1?

   By what number did you multiply both numerator and denominator to change $\frac{1}{6}$ to $\frac{2}{12}$? [2]

2. Have children rename $\frac{1}{4}$ as twelfths. $\left[\frac{1}{4} = \frac{3}{12}\right]$

   Compare both numerators and denominators

   3 is how many times as large as 1? [3 times]
   12 is how many times as large as 4? [3 times]
   By what number do you multiply both terms of $\frac{1}{4}$ to change it to $\frac{3}{12}$?

   Have children multiply both numerator and denominator of $\frac{1}{4}$ by 3.

   $$\frac{1 \times 3}{4 \times 3} = \frac{3}{12}$$

3. Have children rename other fractions:

   $\frac{1}{3}$ as sixths; $\frac{1}{3}$ as ninths; $\frac{2}{3}$ as sixths; $\frac{2}{3}$ as ninths; etc.

By what number did you multiply both numerator and denominator in each case to arrive at the equivalent fraction?

What happens to the value of a fraction when the numerator and denominator are multiplied by the same number, except zero?

(Fundamental Principle of Fractions)

Renaming Fractions: Simple Form

Development

1. Have children:

   Write $\frac{2}{6}$ as thirds; $\frac{3}{9}$ as thirds; $\frac{4}{12}$ as thirds.

   Compare the numerators of $\frac{2}{6}$ and $\frac{1}{3}$. Compare the denominators.

   Compare the numerators of $\frac{3}{9}$ and $\frac{1}{3}$. Compare the denominators.

   Tell how they arrive at $\frac{1}{3}$ from $\frac{2}{6}$; from $\frac{3}{9}$; from $\frac{4}{12}$.

2. Simplify the following and tell by which factor they divided both numerator and denominator.

   Change: $\frac{4}{8}$, $\frac{6}{12}$, $\frac{3}{6}$ to halves Change: $\frac{1}{2}$, $\frac{2}{8}$, $\frac{3}{12}$ fourths

   Change: $\frac{4}{6}$, $\frac{6}{9}$, $\frac{8}{12}$ to thirds Change: $\frac{4}{12}$, $\frac{6}{12}$, $\frac{8}{12}$ to sixths

   Ask children to:

   Reexamine each fractional numeral and its equivalent.
   Discuss what happened to the size of the numerator and denominator after the fraction was changed to its equivalent.
   Discuss what happened to the value of each fraction.
   Tell what you did to the numerator and denominator of each fraction to change to simpler form?
   [Divided numerator and denominator by the same number]

3. Have children change the following to simpler form:

   $\frac{6}{9}=\frac{\square}{n}$ $\left[\frac{2}{3}\right]$ $\frac{12}{16}=\frac{\square}{n}$ $\left[\frac{6}{8}\right]$ $\frac{12}{16}=\frac{\square}{n}$ $\left[\frac{3}{4}\right]$

Ask children:

By what number did you divide the numerator and the denominator of $\frac{6}{9}$ to arrive at $\frac{2}{3}$? [3] of $\frac{12}{16}$ to arrive at $\frac{6}{8}$? [2] of $\frac{12}{16}$ to arrive at $\frac{3}{4}$? [4]

Write the equations as children tell their thinking.

$\frac{6 \div 3}{9 \div 3} = \frac{2}{3}$; $\frac{12 \div 2}{16 \div 2} = \frac{6}{8}$; $\frac{12 \div 4}{16 \div 4} = \frac{3}{4}$

4. Introduce term "common factor".

5. Discuss:

   a. Simplest form of fractions, those having no common factor other than 1 in the numerator and denominator.

   b. The value of the fraction in relation to the size of the parts into which a whole object has been divided.

6. Children state a generalization in their own words for changing a fraction to its simplest form.

*7. Have children observe the fractions $\frac{2}{3}$, $\frac{7}{6}$

For $\frac{2}{3}$, What is the greatest common factor of the numerator and the denominator? [1]

Is $\frac{2}{3}$ in simplest form?

Consider $\frac{7}{6}$. Can we say that $\frac{7}{6}$ is in simplest form as a fraction?

[Yes] Explain.

EVALUATION and / or PRACTICE
SUGGESTED EXERCISES

1. Complete the following statements and tell the common factor used.

$$\frac{1}{3} = \frac{\square}{6}\ ; \qquad \frac{1}{3} = \frac{\square}{9}\ ; \qquad \frac{1}{3} = \frac{\square}{12}\ ; \qquad \frac{3}{4} = \frac{\square}{8}\ ; \qquad \frac{3}{4} = \frac{\square}{12}$$

2. Supply the correct numeral to complete the equations.

$$\frac{1}{6} = \frac{2 \times \square}{2 \times \triangle} \qquad \frac{1}{6} = \frac{3 \times \square}{3 \times \triangle} \qquad \frac{1}{6} = \frac{4 \times \square}{4 \times \triangle}$$

3. Since $\frac{2}{3} = \frac{6}{9}$, then $\frac{2 \times \square}{3 \times \square} = \frac{6}{9}$ Since $\frac{3}{4} = \frac{9}{12}$, then $\frac{3 \times \square}{4 \times \square} = \frac{9}{12}$

   Since $\frac{1}{2} = \frac{4}{8}$, then $\frac{1 \times \square}{2 \times \square} = \frac{4}{8}$ since $\frac{5}{6} = \frac{10}{12}$, then $\frac{5 \times \square}{6 \times \square} = \frac{10}{12}$

4. Complete the following by inserting the correct factor:

$$\frac{2 \div \square}{12 \div \square} = \frac{1}{6} \qquad \frac{3 \div \square}{12 \div \square} = \frac{1}{4} \qquad \frac{4 \div \square}{12 \div \square} = \frac{1}{3}$$

$$\frac{8 \div \square}{12 \div \square} = \frac{2}{3} \qquad \frac{9 \div \square}{12 \div \square} = \frac{3}{4} \qquad \frac{10 \div \square}{12 \div \square} = \frac{5}{6}$$

5. Change the following to simplest form:

$$\frac{2}{6}\ ; \quad \frac{4}{6}\ ; \quad \frac{3}{9}\ ; \quad \frac{6}{9}\ ; \quad \frac{4}{10}\ ; \quad \frac{6}{10}\ ; \quad \frac{8}{10}$$

6. What principle makes it possible to change a fraction to simplest form? Explain, using the problems above. **For example:**

$$\frac{4}{8} = \frac{4 \div 2}{8 \div 2} = \frac{2}{4} = \frac{2 \div 2}{4 \div 2} = \frac{1}{2}$$

$$\frac{4}{8} = \frac{4 \div 4}{8 \div 4} = \frac{1}{2}$$

$$\frac{4}{8} = \frac{1}{2}$$

[In each case numerator and denominator have been divided by the same number]

OPERATIONS

UNIT 41 - MULTIPLICATION OF FRACTIONAL NUMBERS: COMMON FORM

NOTE TO TEACHER

Properties of multiplication that apply to whole numbers also apply to fractional numbers. The properties of Distributivity, Commutativity, and Associativity apply.

Distributivity:

$$4 \times 3\frac{1}{2} =$$

$$4 \times \ 3 + \frac{1}{2} = (4 \times 3) + \left(4 \times \frac{1}{2}\right)$$

$$\boxed{a \times (b + \frac{c}{d}) = (a \times b) + \left(a \times \frac{c}{d}\right)}$$

Commutativity:

$$\frac{2}{3} \times \frac{1}{2} = \frac{1}{2} \times \frac{2}{3}$$

$$\boxed{\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}}$$

Associativity:

$$\left(\frac{3}{4} \times \frac{1}{2}\right) \times \frac{2}{3} = \frac{3}{4} \times \left(\frac{1}{2} \times \frac{2}{3}\right)$$

$$\boxed{\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right)}$$

The Identity Element for multiplication is 1:

$$1 \times \frac{2}{3} = \frac{2}{3} \times 1 = \frac{2}{3}$$

$$\boxed{1 \times \frac{a}{b} = \frac{a}{b}}$$

The Fundamental Property of Fractions

$$\boxed{\frac{a}{b} \times \frac{c}{c} = \frac{ac}{bc} = \frac{a}{b} \qquad (c \neq 0)}$$

One of the interpretations of multiplication of whole numbers is its relationship to repeated addition. (4 x 3 as 3+3+3+3, read as 4 threes or 4 times 3). Similarly one of the interpretations of multiplication of fractions by a whole number is its relationship to repeated addition.

4 times $\frac{2}{3}$ may be interpreted as: $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3}$.

Objectives: To help children develop an understanding of multiplying fractions by whole numbers.

To formulate a generalization for multiplication of fractions and applying that generalization.

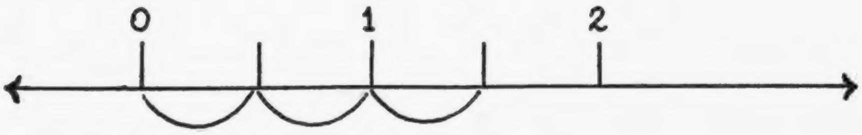
TEACHING SUGGESTIONS

Use problem-solving situations. Children may arrive at solutions "mentally" before using algorithms.

Children should estimate answers and give reasons for their estimates. They should solve these multiplication exercises by using diagrams and repeated addition. Children should be led to realize that repeated addition of fractions can be performed through multiplication by a whole number just as repeated addition of whole numbers can be done through multiplication by a whole number.

1. **Suggested problem:** I need $\frac{1}{2}$ yd. of oilcloth for each of 3 shelves. How much oilcloth do I need for all the shelves?

Estimate: Less than 2 yards.



or

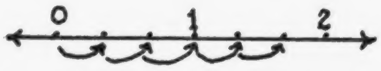
$$\begin{array}{r} \frac{1}{2} \\ \frac{1}{2} \\ \underline{\frac{1}{2}} \\ \frac{3}{2} = 1\frac{1}{2} \end{array}$$

First recorded as: $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} = 1\frac{1}{2}$

2. **Suggested problem:** John wanted to make armbands for the 5 members of the Safety Squad. He required $\frac{1}{3}$ yard of ribbon for each band. How much ribbon did he need?

Children may record the solution of the problem in a variety of ways, including by the use of the number line.

$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{5}{3} = 1\frac{2}{3}$



$$\begin{array}{r} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \underline{\frac{1}{3}} \\ \frac{5}{3} = 1\frac{2}{3} \end{array}$$

Note: Multiplication algorithm should not be used at this time to solve the example; it is only being used to record the solution.

$5 \times \frac{1}{3} = \frac{5}{3} = 1\frac{2}{3}$ (5 x $\frac{1}{3}$ is read as 5 one-thirds or 5 thirds.)

Children may use number line to solve the following:

2 sets of $\frac{1}{6} = n$ 4 sets of $\frac{1}{8} = n$

6 sets of $\frac{1}{4} = n$ 5 sets of $\frac{1}{3} = n$

etc.

3. Suggested problem: A cake recipe calls for $\frac{3}{4}$ of a cup of milk. How much milk do we need to make 5 cakes for our class party.

Estimate: Less than 5 cups; probably less than 4 cups. Why?

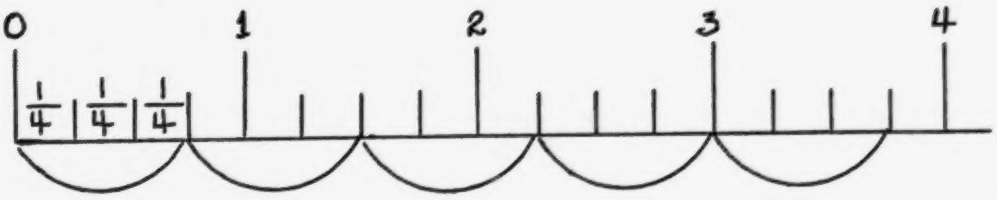
Mental Computation

If each cake needed 1 cup of milk, we would need 5 cups of milk. But each cake needs 1 fourth less. Therefore, 5 cakes need 5 fourths less or 3 and 3 fourths cups.

5 one-fourths are 5 fourths.
5 three-fourths are 15 fourths.

Written Computation

Children may use diagrams and solve by addition.



First recorded as $\frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \frac{15}{4} = 3\frac{3}{4}$

$$\begin{array}{l} \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \hline \frac{15}{4} = 3\frac{3}{4} \end{array}$$

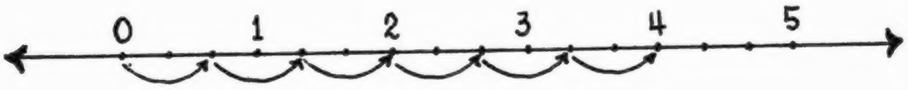
Then recorded as $5 \times \frac{3}{4} = 3\frac{3}{4}$.

Check with estimate.

4. Suggested problem:

$\frac{2}{3}$ of a yard of ribbon was needed for bandoliers for each of the 6 members of the Honor Guard. How much ribbon had to be bought?

$$\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{12}{3} = 4$$



$6 \times \frac{2}{3} = \frac{12}{3} = 4$ ($6 \times \frac{2}{3}$ is read as 6 two-thirds)

Emphasize the recording of the problem as a mathematical sentence involving multiplication.

5. Suggestions for developing a generalization:

Present the following exercises one at a time.
Have children solve each by adding. Then have them examine the two factors and the product.

$2 \times \frac{1}{3} = n \quad \left[\frac{2}{3}\right]$

$2 \times \frac{3}{4} = n \quad \left[\frac{6}{4}\right]$

$3 \times \frac{2}{5} = n \quad \left[\frac{6}{5}\right]$

$6 \times \frac{1}{4} = n \quad \left[\frac{6}{4}\right]$

Ask children: Can you discover a rule for multiplying a fraction by a whole number?

Have children state the generalization in their own words. (Multiplying the numerator of the fraction by the whole number gives the numerator of the product. The denominator remains the same.)

Then solve the following multiplications and verify solutions in various ways.

$n = 6 \times \frac{2}{9} \quad \left[\frac{12}{9} = 1\frac{3}{9} = 1\frac{1}{3}\right]$

$n = 9 \times \frac{2}{5} \quad \left[\frac{18}{5} = 3\frac{3}{5}\right]$

$n = 10 \times \frac{1}{4} \quad \left[\frac{10}{4} = 2\frac{1}{2}\right]$

$n = 8 \times \frac{3}{8} \quad \left[\frac{24}{8} = 3\right]$

EVALUATION and / or PRACTICE
SUGGESTED EXERCISES

1. Record each of the following as a multiplication and multiply.

$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ $\left[4 \times \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \right]$

$\frac{2}{7} + \frac{2}{7} + \frac{2}{7}$ $\left[3 \times \frac{2}{7} = \frac{6}{7} \right]$

Since $4 \times \frac{1}{7} = \frac{4}{7}$

$4 \times \frac{2}{7} = \frac{\square}{7}$ Why? $\left[\frac{2}{7} \text{ is twice } \frac{1}{7} \right]$

Since $4 \times \frac{1}{7} = \frac{4}{7}$

$4 \times \frac{3}{7} = \frac{\square}{7}$ Why?

3. The Connecticut River rose $\frac{1}{3}$ of a foot each hour for 10 hours. How much did the river rise?

4. Susan walked $\frac{3}{5}$ of a mile to the library and discovered that she had forgotten her library card. She walked home to get it and walked back to the library. How far had she walked by the time she reached home with her books?

OPERATIONS

UNIT 42 - FRACTIONAL PARTS OF NUMBERS

Objective: To help children interpret finding a fractional part of a number as multiplication

TEACHING SUGGESTIONS

Finding a Fractional Part of a Number

1. Reinforce the meaning of fractional parts of a group as dividing a group into equal parts (division).

   Have children use pennies, discs, measurement materials if necessary.

   Finding answers in a variety of ways (mental computation) should be emphasized.

   Problems that involve sharing things will be useful for this unit. For example, sharing 10 cookies among 5 children.

2. Reinforce finding fractional parts of numbers through 99.

   Suggested Exercises Using $\frac{1}{2}$, $\frac{1}{4}$, etc. as an Operator.

   Drill:

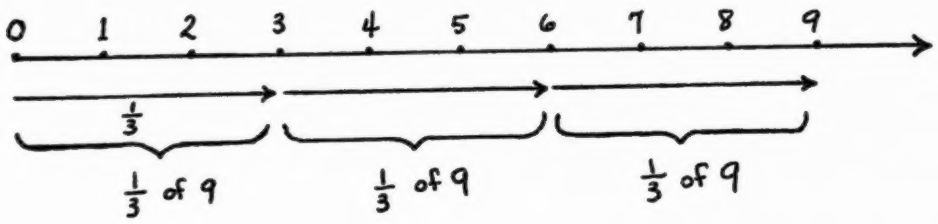
   $\frac{1}{2}$ of numbers through 20, through 100.

   $\frac{1}{4}$ of numbers that are divisible in half, then half again.

   $\frac{1}{8}$ of numbers that are divisible by 8, e.g., 16, 24, 32, 40, etc. (multiples of 8)

   $\frac{1}{3}$ of numbers that are divisible by 3, e.g, 12, 36, 45, etc. (multiples of 3)

Use number rays. For example for $\frac{1}{3}$ of $9 = n$



3. Extend development of $\frac{1}{2}$ as operator to find:

$\frac{1}{2}$ of whole hundreds:

$\frac{1}{2}$ of $200 = n$
$\frac{1}{2}$ of $400 = n$
$\frac{1}{2}$ of $600 = n$
$\frac{1}{2}$ of $800 = n$

$\frac{1}{2}$ of $100 = n$
$\frac{1}{2}$ of $300 = n$
$\frac{1}{2}$ of $500 = n$
$\frac{1}{2}$ of $700 = n$
$\frac{1}{2}$ of $900 = n$

$\frac{1}{2}$ of whole thousands:

$\frac{1}{2}$ of $2000 = n$
$\frac{1}{2}$ of $4000 = n$
$\frac{1}{2}$ of $6000 = n$
$\frac{1}{2}$ of $8000 = n$

$\frac{1}{2}$ of $1000 = n$
$\frac{1}{2}$ of $3000 = n$
$\frac{1}{2}$ of $5000 = n$
$\frac{1}{2}$ of $7000 = n$ etc.

4. Finding one fourth of numbers:

   a. By dividing in half, then half again

      Suggested problem: Find $\frac{1}{4}$ of 600

$\frac{1}{2}$ of 600 = 300; then $\frac{1}{2}$ of 300 = 150

therefore $\frac{1}{4}$ of 600 = 150

or

600 divided into 2 equal parts = 300
300 divided into 2 equal parts = 150
therefore $\frac{1}{4}$ of 600 = 150

$\frac{1}{4}$ of whole hundreds: $\left(\frac{1}{2} \text{ of } \frac{1}{2}\right)$

$\frac{1}{4}$ of 800 = n
$\frac{1}{4}$ of 400 = n
$\frac{1}{4}$ of 600 = n

$\frac{1}{4}$ of 100 = n
$\frac{1}{4}$ of 300 = n
$\frac{1}{4}$ of 500 = n
$\frac{1}{4}$ of 700 = n
$\frac{1}{4}$ of 900 = n

b. By applying the Distributive Property of Multiplication with respect to Addition. For example:

Since 284 = 200 + 80 + 4

$$\frac{1}{4} \text{ of } 284 = \left(\frac{1}{4} \text{ of } 200\right) + \left(\frac{1}{4} \text{ of } 80\right) + \left(\frac{1}{4} \text{ of } 4\right)$$

or

$\frac{1}{4}$ of 284 may be thought through as:

$\frac{1}{4}$ of 200 = 50
$\frac{1}{4}$ of 80 = 20
$\frac{1}{4}$ of 4 = 1

$\frac{1}{4}$ of 284 = 71

$\frac{1}{4}$ of 9000 = n

$$\begin{array}{r} \frac{1}{4} \text{ of } 8000 = 2000 \\ \frac{1}{4} \text{ of } 1000 = 250 \\ \hline \frac{1}{4} \text{ of } 9000 = 2250 \end{array}$$

5. One eighth as an operator

a. Suggested exercise: Find $\frac{1}{8}$ of 300

$\frac{1}{2}$ of 300 = 150
$\frac{1}{2}$ of 150 = 75

Therefore $\frac{1}{4}$ of 300 = 75

$\frac{1}{8} = \frac{1}{2}$ of $\frac{1}{4}$

Therefore $\frac{1}{8}$ of 300 = $\frac{1}{2}$ of 75 = 37 $\frac{1}{2}$

b. One eighth of even hundreds: no fractional remainders

$\frac{1}{8}$ of 1600 = n
$\frac{1}{8}$ of 400 = n
$\frac{1}{8}$ of 200 = n

One eighth of whole hundreds: odd hundreds: fractional remainders:

$\frac{1}{8}$ of 100 = n
$\frac{1}{8}$ of 300 = n
$\frac{1}{8}$ of 500 = n
$\frac{1}{8}$ of 700 = n

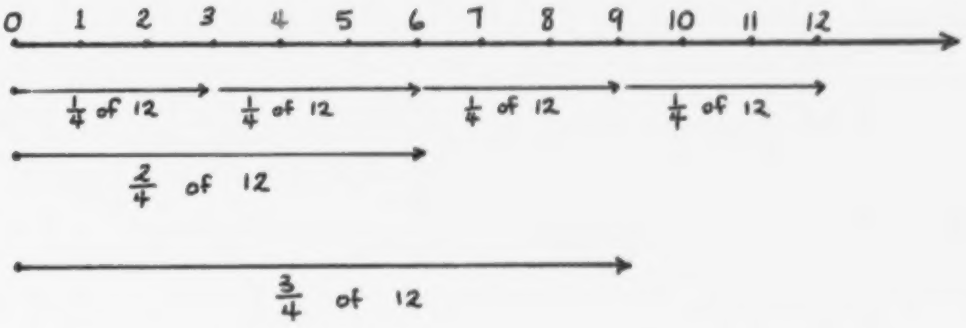
6. Children may find $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{9}$ of numbers in the same way.
For example:

$\frac{1}{3}$ of 7200 = n

$$\begin{array}{l} \frac{1}{3} \text{ of } 6000 = 2000 \\ \frac{1}{3} \text{ of } 1200 = 400 \\ \hline \frac{1}{3} \text{ of } 7200 = 2400 \end{array}$$

7. Finding fractional parts of a number where the numerator of the fraction is greater than 1.

Introduce with a number ray. For example $\frac{3}{4}$ of 12 = n



Discuss with children:

$\frac{2}{4}$ is how many times as great as $\frac{1}{4}$?
$\frac{2}{4}$ of 12 is how many times as great as $\frac{1}{4}$ of 12?
$\frac{3}{4}$ is how many times as great as $\frac{1}{4}$?
$\frac{3}{4}$ of 12 is how many times as great as $\frac{1}{4}$ of 12 ? etc.
Since $\frac{1}{4}$ of 12 = 3, then $\frac{3}{4}$ of 12 = □

Suggested problem: (Related to measures) How many inches are there in $\frac{2}{3}$ of a yard? $\frac{2}{3}$ of 36 = n

$\frac{1}{3}$ of 36 = 12

$\frac{2}{3}$ is twice $\frac{1}{3}$

Therefore $\frac{2}{3}$ of 36 = 2 x 12

$\frac{2}{3}$ of 36 = 24

Have children solve for "n".

$\frac{1}{4}$ of 36 = 9, then $\frac{3}{4}$ of 36 = n
$\frac{1}{8}$ of 56 = 7, then $\frac{7}{8}$ of 56 = n
$\frac{1}{6}$ of 72 =12, then $\frac{5}{6}$ of 72 = n x 12 etc.

8. Suggested exercises for practice

a. $\frac{1}{4}$ of a number = $\frac{1}{2}$ of $\frac{1}{\square}$ of a number.
$\frac{1}{8}$ of a number = $\frac{1}{2}$ of $\frac{1}{\square}$ of a number.
$\frac{1}{8}$ of a number = $\frac{1}{4}$ of $\frac{1}{\square}$ of a number.
$\frac{1}{8}$ of a number = $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{\square}$ of a number.
$\frac{1}{6}$ of a number = $\frac{1}{2}$ of $\frac{1}{\square}$ of a number.
$\frac{1}{6}$ of a number = $\frac{1}{3}$ of $\frac{1}{\square}$ of a number.
$\frac{1}{9}$ of a number = $\frac{1}{3}$ of $\frac{1}{\square}$ of a number.
$\frac{1}{10}$ of a number = $\frac{1}{2}$ of $\frac{1}{\square}$ of a number.
$\frac{1}{10}$ of a number = $\frac{1}{5}$ of $\frac{1}{\square}$ of a number.

b. Complete the following equations to make true statements.

$\frac{1}{2}$ of 100 = n $\frac{1}{4}$ of 100 = n $\frac{1}{2}$ of $\frac{1}{2}$ of 100
$\frac{1}{2}$ of 200 = n etc. $\frac{1}{4}$ of 200 = n etc.
$\frac{1}{2}$ of 1000 = n $\frac{1}{4}$ of 1000 = n
$\frac{1}{2}$ of 2000 = n etc. $\frac{1}{4}$ of 2000 = n etc.

$\frac{1}{8}$ of 100 = n $\frac{1}{2}$ of $\frac{1}{4}$ of 100   or
$\frac{1}{4}$ of $\frac{1}{2}$ of 100
$\frac{1}{8}$ of 1000 = n
$\frac{1}{8}$ of 2000 = n etc.

c. Since $\frac{1}{2}$ of 56 = 28
$\frac{1}{4}$ of 56 = $\frac{1}{2}$ of $\square$
and $\frac{1}{4}$ of 56 = n

Since $\frac{1}{3}$ of 156 = 52
$\frac{1}{6}$ of 156 = $\frac{1}{2}$ of $\square$
and $\frac{1}{6}$ of 156 = n

Since $\frac{1}{2}$ of 128 = 64
$\frac{1}{4}$ of 128 = $\frac{1}{2}$ of $\square$
and $\frac{1}{4}$ of 128 = n

Since $\frac{1}{2}$ of 156 = 78
$\frac{1}{6}$ of 156 = $\frac{1}{3}$ of $\square$
and $\frac{1}{6}$ of 156 = n

d. Find $\frac{1}{4}$ of 300 $\frac{1}{8}$ of 400 $\frac{1}{2}$ of 96 $\frac{5}{6}$ of 60 $\frac{1}{4}$ of 328

e. If $\frac{1}{2}$ of 600 is 300, $\frac{1}{4}$ of 600 is ____; and $\frac{1}{8}$ of 600 is ____.

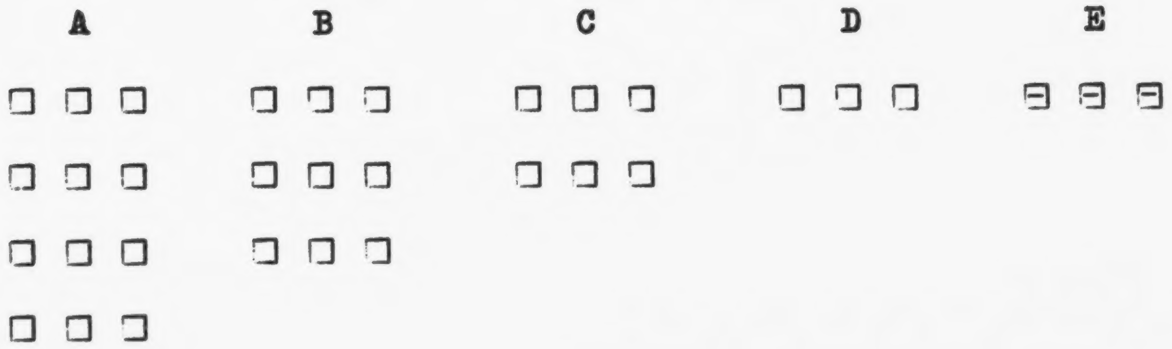
f. John saves $\frac{1}{3}$ of all he earns. One day he earned 96 cents. How much did he save?

g. There are 128 children in Grade 5. $\frac{1}{4}$ of their dental slips have been brought in. How many slips have been brought in?

h. 648 program covers are to be made by the children in Fifth Grade. If Mrs. Jones' class makes $\frac{1}{6}$ of the covers, how many covers will her class make?

Interpretation of Finding a Fractional Part of a Number as Multiplication

1. Have children consider the following diagrams:



How many sets of threes are there in A? [4 sets of the threes]
How many sets of threes are there in B? in C? in D? in E?

As children respond, teacher records horizontally:

4 sets of threes = 12   4 threes = 12 $4 \times 3 = 12$
2 sets of threes = 6   2 threes = 6 $2 \times 3 = 6$
1 set of three = 3   1 three = 3 $1 \times 3 = 3$

2. Ask children to observe the pattern, then to find:

$\frac{1}{2}$ set of threes = $\frac{3}{2}$ $\frac{1}{2}$ three = $\frac{3}{2}$ $\frac{1}{2} \times 3 = \frac{3}{2}$

Have children compare the various ways of expressing the above.

Is 4×3 another way of writing 4 of the threes?
Give another way of recording $\frac{1}{2}$ of 3. $\left[\frac{1}{2} \times 3\right]$

3. Arriving at generalizations

Children should record the following using the "x" sign and solve:

$\frac{1}{4}$ of $3 = \square$ $\frac{1}{5}$ of $2 = \square$ $\frac{1}{3}$ of $9 = \square$

Children then examine $\frac{1}{4} \times 3 = \square$, $\frac{1}{5} \times 2 = \square$, etc. and discuss methods of solution. They should be guided to express the generalization in their own words.

4. Extend the understanding to multiplying a whole number by a fraction whose numerator is greater than 1.

   Record $\frac{2}{3}$ of 12 as $\frac{2}{3} \times 12$; Record $\frac{3}{4}$ of 5 as $\frac{3}{4} \times 5$;

   Record $\frac{5}{8}$ of 72 as $\frac{5}{8} \times 72$.

5. Selecting "of" or "x"

   Children should understand that some times it is more practical to think in terms of "of", and at other times to think in terms of "x".

   e.g. $\frac{3}{4}$ of 20 as $\frac{1}{4}$ of $20 = 5$ or $\frac{3}{4} \times 20 = \frac{60}{4} = 15$

   therefore $\frac{3}{4}$ of $20 = 15$

   Which way do you prefer in each of the following? Why?

   $\frac{2}{3}$ of 12 or $\frac{2}{3} \times 12$

   $\frac{2}{5}$ of 48 or $\frac{2}{5} \times 48$

   $\frac{7}{10}$ of 120 or $\frac{7}{10} \times 120$

   $\frac{3}{8}$ of 16 or $\frac{3}{8} \times 16$

   $\frac{5}{6}$ of 45 or $\frac{5}{6} \times 45$

   $\frac{4}{7}$ of 134 or $\frac{4}{7} \times 134$

Another Algorithm for Finding Fractional Parts of Numbers

1. Reinforce renaming a fraction as a whole number using the Identity Element, 1.

   For example: $\frac{12}{3} = \frac{12 \div 3}{3 \div 3} = \frac{12 \div 3}{1} = 3\overline{)12} = 4$

   Suggested exercises:

$$\frac{24}{8} = \frac{24 \div \square}{8 \div \square} = 24 \div \square = \square\overline{)24} = n$$

$$\frac{150}{25} = \frac{150 \div \square}{25 \div \square} = 150 \div \square = \square\overline{)150} = n$$

$$\frac{2800}{7} = \frac{2800 \div \square}{7 \div \square} = 2800 \div \square = \square\overline{)2800} = n$$

2. Evaluate and /or reinforce finding one half, one third, one fourth, etc. of a number. (Using one half, etc. as the operator on the number.)

Suggested exercises

When we divide a number into 2 equal parts, we are finding one ______ of it.

When we divide a number into 3 equal parts, we are finding one ______ of it.

When we divide a number into 5 equal parts, we are finding one ______ of it.

When we divide a number into 7 equal parts, we are finding one ______ of it.

To find $\frac{1}{7}$ of 294 divide 294 into n equal parts.

3. Discuss as you develop the following:

Problem: There are 168 children in the Fifth Grade. $\frac{1}{4}$ of them are in the Glee Club. How many children are in the Glee Club?

$$\frac{1}{4} \text{ of } 168 = n$$

$$\begin{aligned} \frac{1}{4} \times 168 &= \frac{168}{4} \\ &= \frac{168 \div 4}{4 \div 4} \\ &= \frac{168 \div 4}{1} \\ &= 4\overline{)168} \\ &= 42 \end{aligned}$$

Solution: $n = 42$

42 = 168 divided into 4 equal parts

Problem: The community issued 2200 tickets for the dance festival. Grade 5 sold $\frac{1}{8}$ of the tickets. How many tickets did Grade 5 sell?

$\frac{1}{8}$ of 2200 = n   n = 2200 divided into 8 equal parts.

Answer: Grade 5 sold 275 tickets.

* **Problem:** Grade 6 sold $\frac{3}{8}$ of the 2200 tickets that the Community Association issued for the dance festival. How many tickets did Grade 6 sell?

Refer to the problem suggested for finding $\frac{1}{8}$ of 2200 and discuss:

How you would then find $\frac{3}{8}$ of 2200.

Since $\frac{1}{8}$ of 2200 is $8\overline{)2200}$ or 275, then $\frac{3}{8}$ of 2200 is n x 275

$$\left(\begin{array}{r} 275 \\ \underline{\times\ 3} \\ 825 \end{array}\right)$$

Grade 6 sold 825 tickets.

EVALUATION and / or PRACTICE
SUGGESTED EXERCISES

1. Find: a. $\frac{5}{6}$ of 60 $\qquad$ $\frac{1}{4}$ of 328

   b. $\frac{1}{4}$ divided into 3 parts

   c. 9 divided into thirds = n

2. Mark the example that gives the larger product:

   $3 \times \frac{3}{4}$ or 3×3; $\qquad$ 5×5 or $5 \times \frac{5}{6}$

3. Mother filled $\frac{3}{4}$ of a book of trading stamps. If the book can hold 1200 stamps, how many stamps has she collected?

4. Complete the following:

   $\frac{1}{3} \times 12 = \square$ $\qquad$ $12 \times \frac{1}{3} = \square$ $\qquad$ $\frac{4}{5} \times 15 = \square$ $\qquad$ $15 \times \frac{4}{5} = \square$

5. Verbal problems involving finding fractional parts of numbers may be found in textbooks.